Feedback Stabilization of Parabolic Systems with Input Delay

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Control and analysis of PDE systems

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Outline

Outline

Problem setting

Let consider

$$z' = Az + Bv + f, \quad z(0) = z^0,$$

where

- A is the generator of an analytic semigroup $(e^{tA})_{t\geq 0}$ on a Hilbert space $\mathbb{H}$
- (H) The spectrum of A consists of isolated eigenvalues (λ_j) with finite algebraic multiplicity and there is no finite cluster point in $\{\lambda \in \mathbb{C} \ : \ \mathrm{Re}\,\lambda \geq -\sigma\}$
- $B : \mathbb{U} \to \mathcal{D}(A^*)'$ is a linear operator on a Hilbert space $\mathbb{U}$
- f is a given source satisfying an exponential decay at infinity

Case without delay

- M. Badra, T. Takahashi. 2014: a finite dimensional feedback control is constructed:

A characterization of the exponential stabilization with rate $\sigma > 0$ in the case without delay is the well-known Fattorini-Hautus criterion

$$\forall \varepsilon \in \mathcal{D}(A^*),\ \forall \lambda \in \mathbb{C},\ \operatorname{Re} \lambda \geq -\sigma \quad A^* \varepsilon = \lambda \varepsilon \text{ and } B^* \varepsilon = 0 \\ \Longrightarrow \quad \varepsilon = 0.$$

There exists G with finite rank such that

$$v(t) = G(z(t)), \quad t \geq 0.$$

and v stabilizes the system

$$z' = Az + Bv + f, \quad z(0) = z^0,$$

i.e.

$$\|z(t)\|_{\mathbb{H}} \leq e^{-\sigma t} \left(\|z^0\|_{\mathbb{H}} + \|f(t)\|_{\mathcal{D}(A^*)'} \right), \ \sigma > 0.$$

Problem setting

$$z' = Az + Bv + f, \quad z(0) = z^0,$$

In some cases, due to a calculus time, we can not obtain the value of the state z at the instant t and thus we can not construct a control v as a feedback of z at t.

Objective: obtain a feedback control $v(t)$ that depends on the values of $z(s)$ for $s \leq t - t_0$, where $t_0 > 0$ is a positive constant corresponding to a delay.

Previous results for equations with delay

Consider, for instance, the wave equation with boundary feedback delay:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & x \in (0,L),\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(L,t) = -\alpha u_t(L,t) - \beta u_t(L,t-h), & t > 0, \\ u_t(L,t) = z_0(t), & t \in (-h,0), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in (0,L). \end{cases}$$

Assumption

$$0 \leq \beta < \alpha$$

If it is not the case, it can be shown that instabilities may appear:

- Datko 1988, Datko, Lagnese, Polis 1986 with $\alpha = 0$
- Nicaise, Pignotti 2006 in the more general case for the wave equation (see also Nicaise, V. 2010).

KdV equation with boundary delay

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) & \\ \qquad\qquad + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0,t-h), & t > 0, \\ y_x(0,t) = z_0(t), & t \in (-h,0), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases}$$

Assumption

$$|\alpha| + |\beta| < 1$$

Results [Baudouin, Crépeau, V. 2019]

- Local exponential stability result for non critical lengths L
- The decay rate depends on the delay

KdV equation with internal delay

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + a(x)y(x,t) & \\ \qquad +b(x)y(x,t-h)+y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t>0, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in \omega,\ t \in (-h,0), \end{cases}$$

Results [V. 2021]

- If

$$\exists c_0 > 0, \qquad b(x) + c_0 \leq a(x), \qquad \text{a.e. in } \omega = \operatorname{supp} b$$

  semi-global exponential stability result for all lengths L
- If $\operatorname{supp} b \not\subset \operatorname{supp} a$, local exponential stability for all $L < \sqrt{3}\pi$ and for $\|b\|_{L^\infty(0,L)}$ small enough

Selective bibliography

Several works on the topic

- Krstic 2009 : a backstepping method
- Manitius, Olbrot 1979, Bresch-Pietri, Krstic 2014 : a predictor approach
- Nihtila 1992, Bekiaris-Liberis, Krstic 2013 : case of non constant delay
- Bekiaris-Liberis, Krstic 2017 : multiple delay

Selective bibliography

Our approach inspired from

- Bresch-Pietri, Prieur, Trélat 2018 : finite dimensional linear systems
- Prieur, Trélat 2019 : one-dimensional reaction-diffusion equation with boundary control
- Lhachemi, Shorten 2019 : structurally damped Euler-Bernoulli beam
- Lhachemi, Prieur 2020 : Riesz spectral operator with simple eigenvalues
- Lhachemi, Shorten, Prieur 2020 : control with disturbances and delay depending on time

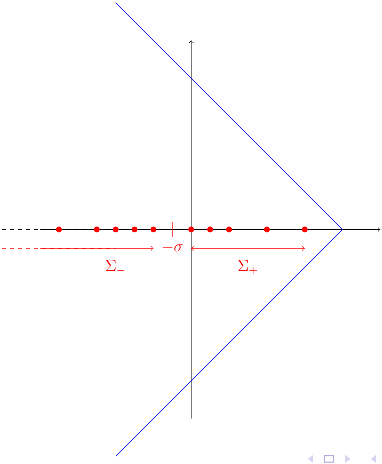
Outline

Construction of time delayed control

Let us consider $\sigma > 0$. We first decompose the spectrum of A into the "unstable" modes and the "stable" modes:

$$\Sigma_+ := \{\lambda_j \ ; \ \mathrm{Re}\,\lambda_j \geq -\sigma\}, \quad \Sigma_- := \{\lambda_j \ ; \ \mathrm{Re}\,\lambda_j < -\sigma\}.$$

Since A is analytic with (H), then Σ_+ is **finite**.



Decomposition of the system

Let consider the projection operator P_+ defined by

$$P_+ := \frac{1}{2\pi\imath} \int_{\Gamma_+} (\lambda - A)^{-1} \, d\lambda.$$

We can define

$$\mathbb{H}_+ := P_+ \mathbb{H}, \quad \mathbb{H}_- := (Id - P_+)\mathbb{H}.$$

$\mathbb{H}_+$ is a finite dimensional space. We have $\mathbb{H}_+ \oplus \mathbb{H}_- = \mathbb{H}$ and if we set

$$A_+ := A_{|\mathbb{H}_+} : \mathbb{H}_+ \to \mathbb{H}_+, \quad A_- := A_{|\mathbb{H}_-} : \mathcal{D}(A) \cap \mathbb{H}_- \to \mathbb{H}_-,$$

then the spectrum of A_+ (resp. A_-) is exactly Σ_+ (resp. Σ_-).

Notation

We can proceed similarly for A^*:

$$P_+^* := \frac{1}{2\pi\imath}\int_{\overline{\Gamma_+}} (\lambda - A^*)^{-1}\, d\lambda,$$

$$\mathbb{H}_+^* := P_+^*\mathbb{H}, \quad \mathbb{H}_-^* := (Id - P_+^*)\mathbb{H},$$

$$A_+^* := A_{|\mathbb{H}_+^*} : \mathbb{H}_+^* \to \mathbb{H}_+^*, \quad A_-^* := A_{|\mathbb{H}_-^*} : \mathcal{D}(A^*) \cap \mathbb{H}_-^* \to \mathbb{H}_-^*.$$

We also define

$$\mathbb{U}_+ := B^*\mathbb{H}_+^*, \quad \mathbb{U}_- := B^*\left(\mathcal{D}(A^*) \cap \mathbb{H}_-^*\right),$$

$$p_+ : \mathbb{U} \to \mathbb{U}_+, \quad p_- : \mathbb{U} \to \mathbb{U}_-, \quad i_+ : \mathbb{U}_+ \to \mathbb{U}, \quad i_- : \mathbb{U}_- \to \mathbb{U},$$

the orthogonal projections and the inclusion maps. We can define

$$B_+ := P_+ B i_+ \in \mathcal{L}(\mathbb{U}_+, \mathbb{H}_+),$$

$$B_- := (Id - P_+) B i_- \in \mathcal{L}(\mathbb{U}_-, \left[\mathcal{D}(A^*) \cap \mathbb{H}_-^*\right]').$$

We can prove that

$$P_+ B = B_+ p_+, \quad (Id - P_+) B = B_- p_-.$$

Decomposition of the system

We set

$$z_+ = P_+ z, \quad z_- = (I - P_+) z$$

z satisfies the system

$$z' = Az + Bv + f, \quad z(0) = z^0,$$

if and only if

$$\begin{cases} z_+' = A_+ z_+ + B_+ p_+ v + P_+ f, & z_+(0) = P_+ z^0, \\ z_-' = A_- z_- + B_- p_- v + (I - P_+) f, & z_-(0) = (I - P_+) z^0. \end{cases}$$

Artstein transform

$$z_+' = A_+ z_+ + B_+ p_+ v + P_+ f, \quad z_+(0) = P_+ z^0$$

The Artstein Transform (Artstein 1982):

$$w(t) := z_+(t) + \int_t^{t+t_0} e^{(t-s)A_+} B_+ p_+ v(s)\, ds.$$

w satisfies the system

$$\begin{cases} w' = A_+ w + e^{-t_0 A_+} B_+ p_+ u + P_+ f, \\ w(0) = z_+(0), \end{cases}$$

such that $u(t) = v(t+t_0)$ for $t \geq 0$.
$(A_+, e^{-t_0 A_+} B_+)$ satisfies the Fattorini-Hautus test?

$$A_+^* \varepsilon = \overline{\lambda_j} \varepsilon, \quad B_+^* e^{-t_0 A_+^*} \varepsilon = 0 \Longrightarrow e^{-t_0 \overline{\lambda_j}} B^* \varepsilon = 0.$$

If (A, B) satisfies the Fattorini's criterion, we get $\varepsilon = 0$.

Stabilization of the closed loop system

Lemma

Let $\sigma > 0$. There exists G of finite rank, such that the solution of

$$\left\{ \begin{array}{c} w' = A_+ w + e^{-t_0 A_+} B_+ G w + P_+ f, \\ w(0) = w^0 \in \mathbb{H}_+, \end{array} \right.$$

satisfies

$$\|w(t)\|_{\mathbb{H}_+} \leq C e^{-\sigma t} \left(\|w^0\|_{\mathbb{H}_+} + \|P_+ f\|_{\mathbb{H}_+} \right), \quad t \geq 0.$$

Thus,

$$v(t) = 1_{[t_0, +\infty)} G(w(t - t_0)).$$

To express the control v in terms of z_+

There exists a kernel $K \in L^\infty_{loc}({\mathbb{R}_+}^2, \mathcal{L}(\mathbb{H}_+))$ such that

$$v(t) = 1_{[t_0,+\infty)}(t)G\left[z_+(t-t_0) + \int_0^{t-t_0} K(t-t_0,s)z_+(s)\ ds\right]$$

Idea of the proof: **Inverse of the Artstein transform**:

$$w(t) := z_+(t) + \int_0^t K(t,s)z_+(s)ds$$

Lemma

Let $D_\infty := \{(t,s) \in \mathbb{R}^2\ :\ t \in (0,\infty),\ s \in (0,t)\}$. *There exists* $K \in L^\infty_{\mathrm{loc}}(D_\infty; \mathcal{L}(\mathbb{H}_+))$ *such that*

$$K(t,s) = e^{(t-s-t_0)A_+}B_+p_+G1_{(\max\{t-t_0,0\},t)}(s)$$
$$+\int_{\max\{t-t_0,s\}}^t e^{(t-\xi-t_0)A_+}B_+p_+GK(\xi,s)\ d\xi \quad (t>0, s\in(0,t)).$$

Proof of the lemma

We set

$$K_0(t) := e^{(t-t_0)A_+} B_+ p_+ G, \quad K_0 \in L^\infty(0, t_0; \mathcal{L}(\mathbb{H}_+)),$$

so the equation writes

$$K(t,s) = K_0(t-s) 1_{(\max\{t-t_0,0\},t)}(s) + \int_{\max\{t-t_0,s\}}^t K_0(t-\xi) K(\xi,s)\, d\xi.$$

Let $T > 0$, and let us define

$$D_T = \{(t,s) \in \mathbb{R}^2 \mid \quad t \in (0,T), \quad s \in (0,t)\},$$

and

$$\Phi : L^\infty(D_T; \mathcal{L}(\mathbb{H}_+)) \to L^\infty(D_T; \mathcal{L}(\mathbb{H}_+)),$$

$$(\Phi K)(t,s) = \int_{\max\{t-t_0,s\}}^t K_0(t-\xi) K(\xi,s)\, d\xi, \quad ((t,s) \in D_T).$$

The mapping Φ is well-defined, and is a linear and bounded operator of $L^\infty(D_T; \mathcal{L}(\mathbb{H}_+))$. Moreover,

$$\|(\Phi K)(t,s)\|_{\mathcal{L}(\mathbb{H}_+)} \leq t \, \|K_0\|_{L^\infty(0,t_0;\mathcal{L}(\mathbb{H}_+))} \, \|K\|_{L^\infty(D_T;\mathcal{L}(\mathbb{H}_+))} .$$

Proof of the lemma

$$\left\|(\Phi^2 K)(t,s)\right\|_{\mathcal{L}(\mathbb{H}_+)} = \left\| \int_{\max\{t-t_0,s\}}^{t} K_0(t-\xi)\Phi K(\xi,s)\ d\xi \right\|_{\mathcal{L}(\mathbb{H}_+)}$$

$$\leq \|K_0\|_{L^\infty(0,t_0;\mathcal{L}(\mathbb{H}_+))} \int_{\max\{t-t_0,s\}}^{t} \|\Phi K(\xi,s)\|_{\mathcal{L}(\mathbb{H}_+)}$$

$$\leq \|K_0\|^2_{L^\infty(0,t_0;\mathcal{L}(\mathbb{H}_+))} \|K\|_{L^\infty(D_T;\mathcal{L}(\mathbb{H}_+))} \int_{\max\{t-t_0,s\}}^{t} \xi d\xi$$

$$\leq \frac{t^2}{2} \|K_0\|^2_{L^\infty(0,t_0;\mathcal{L}(\mathbb{H}_+))} \|K\|_{L^\infty(D_T;\mathcal{L}(\mathbb{H}_+))},$$

and by induction

$$\left\|(\Phi^n K)(t,s)\right\|_{\mathcal{L}(\mathbb{H}_+)} \leq \frac{t^n}{n!} \|K_0\|^n_{L^\infty(0,t_0;\mathcal{L}(\mathbb{H}_+))} \|K\|_{L^\infty(D_T;\mathcal{L}(\mathbb{H}_+))}.$$

End of the proof of the lemma

In particular, for n large enough, Φ^n is a strict contraction and consequently if we define $\widetilde{\Phi}$ by

$$(\widetilde{\Phi}K)(t,s) := (\Phi K)(t,s) + K_0(t-s)1_{(\max\{t-t_0,0\},t)}(s)$$

then $\widetilde{\Phi}^n$ is also a strict contraction. This implies that $\widetilde{\Phi}$ admits a unique fixed point.

Remark

As $rank(G) \leq N_+$, we can write

$$G(\phi) = \sum_{k=1}^{N_+} (\phi, \zeta_k)_{\mathbb{H}}\, v_k, \quad (\phi \in \mathbb{H}_+),$$

where $v_k \in \mathbb{U}_+$, $\zeta_k \in \mathbb{H}_+^*$.

End of the proof of the main result

I remains:

- to prove that this feedback stabilizes the whole system: it does not destabilize the infinite dimensional system that verifies z_-

$$z_-' = A_- z_- + B_- p_- v + (I - P_+) f, \quad z_-(0) = (I - P_+) z^0.$$

There exists $\sigma_- > \sigma$ such that

$$\left\| e^{A_- t} \right\|_{\mathcal{L}(\mathbb{H}_-)} \leq C e^{-\sigma_- t}, \quad \left\| (\lambda_0 - A)^\gamma e^{A_- t} \right\|_{\mathcal{L}(\mathbb{H}_-)} \leq C \frac{1}{t^\gamma} e^{-\sigma_- t}.$$

For $t \geq t_0$,

$$z_-(t) = e^{A_- t}(Id - P_+) z^0 + \int_{t_0}^t (\lambda_0 - A)^\gamma e^{A_-(t-s)} (\lambda_0 - A)^{-\gamma} B_- p_- G w(s - t_0)\, ds$$
$$+ \int_0^t e^{A_-(t-s)} (Id - P_+) f(s)\, ds.$$

$$\Longrightarrow \|z_-(t)\|_{\mathbb{H}} \leq C e^{-\sigma_- t} \left\| z^0 \right\|_{\mathbb{H}}$$
$$+ C e^{-\sigma t} \int_{t_0}^t \frac{1}{(t-s)^\gamma} e^{-(\sigma_- - \sigma)(t-s)}\, ds \left(\|P_+ z^0\|_{\mathbb{H}_+} + \|P_+ f\|_{L^2_\sigma(0,\infty;\mathbb{H}_+)} \right)$$
$$+ C e^{-\sigma t} \int_0^t \frac{1}{(t-s)^{\gamma'}} e^{-(\sigma_- - \sigma)(t-s)} \|e^{\sigma s} f(s)\|_{\mathbb{H}_{-\gamma'}}\, ds$$

Main result

Using that $\sigma_- > \sigma$, $\gamma < 1$ and $\gamma' < 1/2$, we deduce that

$$\|z_-(t)\|_{\mathbb{H}_-} \leq Ce^{-\sigma t} \left(\|z^0\|_{\mathbb{H}} + \|f\|_{L^2_\sigma(0,\infty;\mathbb{H}_{-\gamma'})} \right).$$

Theorem

Assume that (A, B) satisfies the Fattorini criterion. There exists a kernel $K \in L^\infty_{loc}({\mathbb{R}_+}^2, \mathcal{L}(\mathbb{H}_+))$, $\zeta_k \in \mathcal{D}(A^*)$, $v_k \in B^* (\mathcal{D}(A^*))$, $k = 1, \ldots, N_+$, such that

$$v(t) = 1_{[t_0,+\infty)}(t) \sum_{k=1}^{N_+} \left(z(t-t_0) + \int_0^{t-t_0} K(t-t_0, s) z(s) \, ds, \zeta_k \right)_{\mathbb{H}} v_k,$$

stabilizes the whole system

$$z' = Az + Bv + f, \quad z(0) = z^0,$$

Reminder of the ideas of the proof

$\boxed{\text{Fattorini-Hautus criterion for } (A,B)}$

$$\big(A^*\epsilon = \lambda\epsilon \text{ for } \lambda \in \Sigma_+ \text{ and } B^*\epsilon = 0\big) \quad \Longrightarrow \quad \epsilon = 0$$

$$\Downarrow$$

$\boxed{\text{Exponential stabilization for the system that satisfies } w}$

$$\Downarrow$$

$\boxed{\text{Exponential stabilization for the system that satisfies } z_+}$

$$\Downarrow$$

$\boxed{\text{Exponential stabilization for the whole system that satisfies } z}$

Outline

Heat equation

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain of class $C^{2,1}$ and let $\mathcal{O} \subset \Omega$.

$$\begin{cases} \partial_t z - \Delta z = v 1_{\mathcal{O}} & \text{in } (0,\infty) \times \Omega, \\ z = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{cases}$$

$$\mathbb{H} = L^2(\Omega), \quad \mathbb{U} = L^2(\mathcal{O}),$$

$$Az = \Delta z, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad A = A^*.$$

$$Bv = v 1_{\mathcal{O}}.$$

If ε satisfies $A^*\varepsilon = \lambda\varepsilon$ and $B^*\varepsilon = 0$, then

$$\begin{cases} \lambda\varepsilon - \Delta\varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \varepsilon = 0 & \text{in } \mathcal{O}. \end{cases}$$

From standard results on the unique continuation of the Laplace operator, we deduce that $\varepsilon = 0$.

Reaction-convection-diffusion equations

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain of class $C^{2,1}$ and Γ a non-empty open subset of $\partial\Omega$.

$$\begin{cases} \partial_t z - \Delta z - b \cdot \nabla z - cz = 0 & \text{in } (0,\infty) \times \Omega, \\ z = v & \text{on } (0,\infty) \times \Gamma, \\ z = 0 & \text{on } (0,\infty) \times (\partial\Omega \setminus \Gamma), \\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{cases}$$

$$\mathbb{H} = L^2(\Omega), \quad \mathbb{U} = L^2(\Gamma),$$

$$Az = \Delta z + b \cdot \nabla z + cz, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

$$\mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega), \quad A^*\varepsilon = \Delta\varepsilon - \overline{b} \cdot \nabla\varepsilon + (\overline{c - \mathrm{div} b})\varepsilon,$$

$$B^*\varepsilon = -\frac{\partial\varepsilon}{\partial\nu}_{|\Gamma}.$$

If ε satisfies $A^*\varepsilon = \lambda\varepsilon$ and $B^*\varepsilon = 0$, then

$$\begin{cases} \lambda\varepsilon - \Delta\varepsilon + \overline{b} \cdot \nabla\varepsilon - (\overline{c - \mathrm{divb}})\varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial\varepsilon}{\partial\nu} = 0 & \text{on } \Gamma. \end{cases}$$

From standard results on the unique continuation of the Laplace operator, we deduce that $\varepsilon = 0$.

Oseen system

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,1}$ and let $\mathcal{O} \subset \Omega$.

$$\left\{\begin{array}{rl} \partial_t z + (w^S \cdot \nabla) z + (z \cdot \nabla) w^S - \nu \Delta z + \nabla q = 1_{\mathcal{O}} v & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot z = 0 & \text{in } (0,\infty) \times \Omega, \\ z = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{array}\right.$$

$\mathbb{H} = \{z \in [L^2(\Omega)]^3 \ : \ \nabla \cdot z = 0 \text{ in } \Omega,\ z \cdot n = 0 \text{ on } \partial\Omega\}, \quad \mathbb{U} = [L^2(\mathcal{O})]^3.$

We denote by $\mathbb{P}$ the orthogonal projection $\mathbb{P} : [L^2(\Omega)]^3 \to \mathbb{H}$.

$\mathcal{D}(A) = [H^2(\Omega) \cap H^1_0(\Omega)]^3 \cap \mathbb{H}, \quad Az = \mathbb{P}\left(\nu \Delta z - (w^S \cdot \nabla) z - (z \cdot \nabla) w^S\right)$

$$Bv = \mathbb{P}\left(1_{\mathcal{O}} v\right), \quad B^* \varepsilon = \varepsilon_{|\mathcal{O}}$$

If ε satisfies $A^* \varepsilon = \lambda \varepsilon$ and $B^* \varepsilon = 0$, then

$$\left\{\begin{array}{rl} \lambda \varepsilon - \nu \Delta \varepsilon - (w^S \cdot \nabla) \varepsilon + (\nabla w^S)^* \varepsilon + \nabla \pi = 0 & \text{in } \Omega, \\ \nabla \cdot \varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \varepsilon \equiv 0 & \text{in } \mathcal{O}. \end{array}\right.$$

By [Fabre Lebeau 1996], we deduce that $\epsilon = 0$.

Navier-Stokes system

We consider the stabilization of the Navier-Stokes system with internal control:

$$\left\{ \begin{array}{rl} \partial_t \widetilde{z} + (\widetilde{z} \cdot \nabla)\widetilde{z} - \nu \Delta \widetilde{z} + \nabla \widetilde{q} = 1_{\mathcal{O}} v + f^S & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot \widetilde{z} = 0 & \text{in } (0,\infty) \times \Omega, \\ \widetilde{z} = b^S & \text{on } (0,\infty) \times \partial\Omega, \\ \widetilde{z}(0,\cdot) = \widetilde{z}^0 & \text{in } \Omega, \end{array} \right.$$

around the stationary state

$$\left\{ \begin{array}{rl} (w^S \cdot \nabla) w^S - \nu \Delta w^S + \nabla r^S = f^S & \text{in } \Omega, \\ \nabla \cdot w^S = 0 & \text{in } \Omega, \\ w^S = b^S & \text{on } \partial\Omega. \end{array} \right.$$

We obtain the local stabilization result for the Navier-Stokes system with internal control with delay.

Conclusion

Conclusion

- Large class of parabolic systems with input delay
- The Fattorini-Hautus criterion yields the existence of such a feedback control, as in the case of stabilization without delay
- Application to several systems

Open problems

- Time dependent delay
- Presence of disturbances

Happy birthday Marius !