Control and analysis of PDE systems

Institut de Mathématiques Bordeaux

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to celebrate the 60th birthday of Marius Tucsnak

Approximation of feedback gains stabilizing fluid flows using the pseudo-compressibility method

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joint work with Mehdi Badra

Marius...

Marius...

- I met Marius for the first time in 1997.

- Next, I met Marius in conferences, seminars, projects ... (with of without J.-P. ?) in
- Nancy, Paris, Pont à Mousson, Bordeaux, ...Point à Pitre ...
- Shangai, Vancouver, Zurich, San Diego, somewhere in Portugal, in Marocco, Roma (?), Cortona (?), Bangalore, Tunis, Oberwolfach
- Vorau, Graz, Linz, Craiova ...

- I learned a lot from discussions with Marius, from his papers...
- Fine results in Control Theory
- Fluid-structure-interaction problems
- Riccati equations. Mistakes by some colleagues...

- K. Le Balc'h, M. Tucsnak, A penalty approach to the infinite horizon LQR optimal control problem for the linearized Boussinesq system. ESAIM Control Optim. Calc. Var. 27 (2021).

Outline of the talk

Part I

- Approximation of the Oseen and Boussinesq systems by the pseudo-compressible method (also called the penalty method - Temam 66)

- Brief review on the pseudo-compressibility method. Known results in approximation and control

Part II

- Distributed stabilization of the Oseen and Boussinesq systems using their pseudo-compressible approximation

Part III

- Boundary stabilization of the Oseen and Boussinesq systems using their pseudo-compressible approximation

- Ω is either a bounded domain in $\mathbb{R}^3$ either of class C^2, or a bounded polyhedral convex domain.
- $(v_s, q_s) \in (H^1(\Omega))^3 \times L^2_0(\Omega)$ is a stationary solution of the N.S.E:

$$(v_s \cdot \nabla) v_s - \nu \, \Delta v_s + \nabla q_s = f_s, \quad \operatorname{div} v_s = 0 \quad \text{in} \quad \Omega,$$
$$v_s = g_s \quad \text{on} \quad \Gamma = \partial\Omega.$$

- The control Navier-Stokes system

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v - \nu \, \Delta v + \nabla q = f_s + \chi_{\mathcal{O}} \, u, \quad \text{in} \quad Q = \Omega \times (0, \infty),$$
$$\operatorname{div} u = 0 \quad \text{in} \quad Q, \quad u = g_s \quad \text{on} \quad \Sigma = \Gamma \times (0, \infty),$$
$$u(0) = u_0 = v_s + y_0 \quad \text{in} \quad \Omega,$$

y_0 is a perturbation in the I.C.

The Oseen system

The nonlinear system satisfied by $(y, q) = (v, q) - (v_s, q_s)$ is

$$\frac{\partial y}{\partial t} + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s + \kappa (y \cdot \nabla) y - \nu \, \Delta y + \nabla q = \chi_{\mathcal{O}} \, u \quad \text{in} \quad Q,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad Q,$$

$$y = 0 \quad \text{on} \quad \Sigma,$$

$$y(0) = y_0 \quad \text{in} \quad \Omega,$$

with $\kappa = 1$. The associated linearized system is obtained by setting $\kappa = 0$.

The stationary Boussinesq system

The stationary Boussinesq system

$$(v_s \cdot \nabla) v_s - \nu \Delta v_s + \nabla q_s = f_s + \vec{\beta} \tau_s, \quad \operatorname{div} v_s = 0 \quad \text{in} \quad \Omega,$$

$$v_s = h_s \quad \text{on} \quad \Gamma.$$

$$-\mu \Delta \tau_s + v_s \cdot \nabla \tau_s = g_s, \quad \text{in} \quad \Omega, \quad \tau_s = k_s \quad \text{on} \quad \Gamma,$$

with $\nu > 0$, $\mu > 0$, and $\vec{\beta}$ is a vector in $\mathbb{R}^3$. We assume that a variational solution $(v_s, q_s, \tau_s) \in (H^1(\Omega))^3 \times L^2(\Omega) \times H^1(\Omega)$ exists.

The nonlinear Boussinesq system

We consider the control Boussinesq system

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v - \nu \Delta v + \nabla q = f_s + \vec{\beta}\,\tau + \chi_{\mathcal{O}}\, u_y, \quad \text{in } Q = \Omega \times (0, \infty),$$

$$\operatorname{div} v = 0 \quad \text{in } Q, \quad v = h_s \quad \text{on } \Sigma = \Gamma \times (0, \infty),$$

$$v(0) = v_0 = v_s + y_0 \quad \text{on} \quad \Omega,$$

$$\frac{\partial \tau}{\partial t} - \mu \Delta \tau + v \cdot \nabla \tau = g_s + \chi_{\mathcal{O}}\, u_\theta, \quad \text{in } Q,$$

$$\tau = k_s \quad \text{on} \quad \Gamma,$$

$$\tau(0) = \tau_0 = \tau_s + \theta_0 \quad \text{on} \quad \Omega.$$

In the above system $u_y \in L^2(0, \infty; (L^2(\Omega))^3)$ and $u_\theta \in L^2(0, \infty; L^2(\Omega))$ are the control variables. We set $u = \begin{pmatrix} u_y \\ u_\theta \end{pmatrix}$.

The linearized Boussinesq system

The nonlinear system satisfied by $(y, p, \theta) = (v, q, \tau) - (v_s, q_s, \tau_s)$ is

$$\frac{\partial y}{\partial t} + (y \cdot \nabla) v_s + (v_s \cdot \nabla) y + \kappa (y \cdot \nabla) y - \nu \Delta y + \nabla p = \vec{\beta}\, \theta + \chi_{\mathcal{O}}\, u_y,$$

$$\operatorname{div} y = 0 \text{ in } Q, \quad y = 0 \text{ on } \Sigma, \quad y(0) = y_0 \text{ in } \Omega,$$

$$\frac{\partial \theta}{\partial t} - \mu \Delta \theta + y \cdot \nabla \tau_s + v_s \cdot \nabla \theta + \kappa\, y \cdot \nabla \theta = \chi_{\mathcal{O}}\, u_\theta, \quad \text{in } Q,$$

$$\theta = 0 \quad \text{on} \quad \Sigma, \quad \theta(0) = \theta_0 \quad \text{on} \quad \Omega,$$

with $\kappa = 1$. The Boussinesq system linearized around (v_s, τ_s) corresponds to $\kappa = 0$.

Approximation by the pseudo-compressible model

- Pseudo-compressible approximation

$$\frac{\partial y_\varepsilon}{\partial t} - \nu \Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) y_\varepsilon + \nabla p_\varepsilon = \chi_{\mathcal{O}}\, u \ \text{ in } Q,$$

$$\operatorname{div} y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q, \qquad y_\varepsilon = 0 \ \text{ on } \Sigma, \qquad y_\varepsilon(0) = y_0 \ \text{ in } \Omega.$$

v_s^ε is an approximation of v_s.

- The equation for y_ε can be solved first

$$\frac{\partial y_\varepsilon}{\partial t} - \nu \Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) y_\varepsilon - \frac{1}{\varepsilon} \nabla \operatorname{div} y_\varepsilon = \chi_{\mathcal{O}}\, u \ \text{ in } Q.$$

- Find u in feedback form $u = Ky$ ($u_\varepsilon = K_\varepsilon y_\varepsilon$) able to stabilize the incomp. model (resp. pseudo-compressible model).
- Study convergence results $K_\varepsilon \to K$, $y_\varepsilon \to y$.
- Prove that the feedback $K_\epsilon P$ also stabilizes the original system.

Error estimates for the pseudo-compressible approximation

$$-\nu\Delta v + \nabla p = f \ \text{ in } Q,$$
$$\operatorname{div} v = 0 \text{ in } \Omega, \qquad v = 0 \ \text{ on } \Gamma.$$

- The pseudo-compressible Stokes system

$$-\nu\Delta v_\varepsilon + \nabla p_\varepsilon = f \ \text{ in } Q,$$
$$\operatorname{div} v_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } \Omega, \qquad v_\varepsilon = 0 \ \text{ on } \Gamma.$$

- Temam (1977), Bercovier (1978) - Approximation error for Stokes

$$\|v - v_\varepsilon\|_{(H^1(\Omega))^3} + \|p - p_\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon\,\|f\|_{(H^{-1}(\Omega))^3}.$$

- Temam (1977), Hebecker (1982), Shen (1995)

$$\|v - v_\varepsilon\|_{L^2(H^1(\Omega))\cap L^\infty(L^2(\Omega))} + \|\operatorname{div} v_\varepsilon\|_{L^\infty(L^2(\Omega))}$$
$$\leq C\,\varepsilon^{1/2}\left(\|f\|_{L^2((H^{-1}(\Omega))^3)} + \|y_0\|_{(L^2(\Omega))^3}\right).$$

Known results in control

- Chrysafinos (2004) - LQR problem for Stokes, with a distributed control, with a finite time horizon.

- Badra, Buchot, Thevenet (2011) LQR problem for Oseen, with a boundary control, with a finite time horizon.

- Le Balc'h, Tucsnak (2021) LQR problem for Boussinesq, with a distributed control, with an infinite time horizon. Feedback stabilization + Null controllability.

Boussinesq: $z' = Az + Bu, \quad z(0) = z_0 \in Z = V_n^0(\Omega) \times L^2(\Omega),$

p-c Boussinesq: $z_\varepsilon' = A_\varepsilon z_\varepsilon + B_\varepsilon u, \quad z(0) \in Z_\varepsilon = (L^2(\Omega))^4,$

Riccati based feedbacks: $(A, B, I), \quad \int_0^\infty (\|u(t)\|_U^2 + \|z(t)\|_H^2)dt,$

$U = (L^2(0, \infty; L^2(\Omega)))^4, \quad H = (L^2(\Omega))^4.$

Results proved by Kevin Le Balc'h and Marius Tucsnak

- Uniform (in ε) global Carleman inequality (for p-c Boussinesq): By combining a global Carleman inequality for the p-c Oseen system, with $v_s \in (W^{1,\infty}(\Omega))^3$ (Badra-2011), and a global Carleman inequality for the heat equation with $\tau_s \in W^{1,\infty}(\Omega)$.

- Null controllability result for the p-c Boussinesq system, with control cost unif. in $\varepsilon \in (0, \varepsilon_0)$.

- Unif. exp. feed. stab. Convergence of Riccati-based feedbacks: For all $z_0 = (y_0, \theta_0) \in Z = V_n^0(\Omega) \times L^2(\Omega)$,

$$\lim_{\varepsilon \to 0} \|(\Pi_\varepsilon - \Pi) z_0\|_H = 0,$$

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|e^{t(A_\varepsilon - B_\varepsilon \Pi_\varepsilon)} z_0 - e^{t(A - B\Pi)} z_0\|_H = 0,$$

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_{\mathrm{opt},\varepsilon}(t) - u_{\mathrm{opt}}(t)\|_U = 0,$$

for all $T > 0$.

The control Oseen system

$$\frac{\partial y}{\partial t} + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s - \nu \, \Delta y + \nabla q = \chi_{\mathcal{O}} \, u \quad \text{in} \quad Q,$$
$$\operatorname{div} y = 0 \quad \text{in} \quad Q, \quad y = 0 \quad \text{on} \quad \Sigma, \quad y(0) = y_0 \quad \text{in} \quad \Omega.$$

The Leray projector $P \in \mathcal{L}(H, Z)$, $H = (L^2(\Omega))^3$, $Z = V_n^0(\Omega)$.

$V_n^0(\Omega) = \{y \in L^2(\Omega; \mathbb{R}^3) \mid \operatorname{div} y = 0, \ y \cdot n = 0 \text{ on } \Gamma\}$

The Oseen operator $(A, \mathcal{D}(A))$, and the control op. B

$Ay = P(\nu \, \Delta y - (v_s \cdot \nabla) y - (y \cdot \nabla) v_s)$,
$\mathcal{D}(A) = V_n^0(\Omega) \cap (H_0^1(\Omega) \cap H^2(\Omega))^3, \quad B = P\chi_{\mathcal{O}}.$

The control Oseen system

$$y' = Ay + Bu, \quad y(0) = y_0.$$

The control Oseen system

The stationary solution (v_s, q_s) belongs to $(H^1(\Omega))^3 \times L^2(\Omega)$.
For all w_s satisfying the H^1-bound

$$\|w_s\|_{(H^1(\Omega))^3} \leq \|v_s\|_{(H^1(\Omega))^3} + 1,$$

we set

$$A_{w_s} z = P(\nu \Delta z - (z \cdot \nabla) w_s - (w_s \cdot \nabla) z), \quad \mathcal{D}(A_{w_s}) = \mathcal{D}(A),$$

and

$$a_{w_s}(z, \zeta) = \int_\Omega (\nu \nabla z : \nabla \zeta + (w_s \cdot \nabla) z \cdot \zeta + (z \cdot \nabla) w_s \cdot \zeta) \, dx,$$

for all $z \in (H^1(\Omega))^3$, $\zeta \in (H^1(\Omega))^3$.

Uniform analyticity

We can choose $\omega_0 > 0$ such that

$$\omega_0 \|z\|^2_{(L^2(\Omega))^3} + a_{w_s}(z,z) \geq \frac{\nu}{2} \|z\|^2_{(H^1(\Omega))^3},$$

for all $z \in (H^1(\Omega))^3$ and all w_s satisfying the H^1-bound.

For all w_s satisfying the H^1-bound, the operator $(A_{w_s}, \mathcal{D}(A_{w_s}))$ is the infinitesimal generator of an analytic semigroup on $Z = V_n^0(\Omega)$. There exists a sector $\{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$, with $\delta \in]0, \pi/2[$, such that

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_{w_s}),$$
$$\|(\lambda I - A_{w_s})^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\lambda - \omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta},$$

for all w_s satisfying the H^1-bound.

The pseudo-compressible control Oseen system

We assume that $\|v_s^\varepsilon - v_s\|_{(H^1(\Omega))^3} \leq C_s\,\varepsilon, \quad \forall \varepsilon \in (0,1)$.
We set $\varepsilon_0 = 1/C_s$. The pseudo-compressible Oseen operator A_ε is

$$\mathcal{D}(A_\varepsilon) = (H^2(\Omega) \cap H_0^1(\Omega))^3,$$
$$A_\varepsilon v = \nu \Delta v - (v \cdot \nabla) v_s^\varepsilon - (v_s^\varepsilon \cdot \nabla) v + \tfrac{1}{\varepsilon} \nabla(\operatorname{div} v).$$

The pseudo-compressible system can be rewritten in the form

$$y_\varepsilon' = A_\varepsilon y_\varepsilon + B_\varepsilon u, \quad y_\varepsilon(0) = y_0, \quad \text{with } B_\varepsilon = \chi_{\mathcal{O}}.$$

For all $\varepsilon \in (0, \varepsilon_0)$, the operator $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ is the infinitesimal generator of an analytic semigroup on $(L^2(\Omega))^3$. We have

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_\varepsilon),$$
$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C}{|\lambda - \omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta},$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Stabilizability of the pair (A, B) in Z

- Boussinesq: The stationary solution (v_s, q_s, τ_s) satisfies

$$v_s \in (H^1(\Omega) \cap L^\infty(\Omega))^3 \quad \text{and} \quad \tau_s \in H^1(\Omega) \cap L^\infty(\Omega).$$

Under that assumption, the pair (A, B) is stabilizable in $Z = V_n^0(\Omega) \times L^2(\Omega)$. That follows from the unique continuation result

$$(\phi, \xi) \in \mathcal{D}(A^*),\ \lambda \in \mathbb{C}, \quad A^* \begin{pmatrix} \phi \\ \xi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \xi \end{pmatrix},$$

$$\text{with} \quad \phi = 0 \quad \text{and} \quad \xi = 0, \quad \text{in } \mathcal{O},$$

$$\text{obeys} \quad \phi = 0 \quad \text{and} \quad \xi = 0, \quad \text{in } \Omega.$$

- For Oseen. Local Carleman estimate for Oseen, Badra-Takahashi, 2014.

- For Boussinesq. Rewriting the adjoint system + combined with local Carleman estimates, 2021.

Stabilizability of the pair $(A_\varepsilon, B_\varepsilon)$

From the exponential stability of $\left(e^{t(A+\omega_K I+BK)}\right)_{t\geq 0}$, with $\omega_K > 0$ and $K \in \mathcal{L}(Z, U)$, we deduce that of $(A_\varepsilon, B_\varepsilon)$, provided that:

- Analytic estimate for $(A, \mathcal{D}(A))$ and uniform analytic estimate for $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$

- Convergence rate of A_ε towards A (with $\lambda_0 > \omega_0$)

$$\|(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \leq C\varepsilon^s, \quad \forall \varepsilon \in (0, \varepsilon_0), \quad s > 0.$$

- Uniform bounds for $P_\varepsilon \in \mathcal{L}(H)$ and $B_\varepsilon \in \mathcal{L}(U, Z_\varepsilon)$.

- Convergence rate of B_ε towards B

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}(U,H)} \leq C\varepsilon^r \quad \forall \varepsilon \in (0, \varepsilon_0), \quad 0 < r \leq s.$$

ε_0 can be chosen such that $\left(e^{t(A_\varepsilon+\omega_{K,\varepsilon} I+B_\varepsilon K)}\right)_{t\geq 0}$ is exponentially stable, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, with $\omega_{K,\varepsilon} = \omega_K - \rho\varepsilon^r$.

The pseudo-compressible control Oseen system

- The following bounds hold, uniformly in $\varepsilon \in (0, \varepsilon_0)$:

$$\|z\|_{(H^2(\Omega))^3} + \frac{1}{\varepsilon}\|\mathrm{div}\, z\|_{H^1(\Omega)} \leq C\|(\lambda_0 I - A_\varepsilon)z\|_{(L^2(\Omega))^3}, \ \forall z \in \mathcal{D}(A_\varepsilon),$$
$$\|\phi\|_{(H^2(\Omega))^3} + \frac{1}{\varepsilon}\|\mathrm{div}\, \phi\|_{H^1(\Omega)} \leq C\|(\lambda_0 I - A_\varepsilon^*)\phi\|_{(L^2(\Omega))^3}, \ \forall \phi \in \mathcal{D}(A_\varepsilon^*).$$

- The following approximation property holds:

$$\|(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}\|_{\mathcal{L}((L^2(\Omega))^3)} \leq C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

- The control operators B and B_ε satisfy

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}((L^2(\Omega))^3)} \leq C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

$$(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon = [(\lambda_0 I - A)^{-1} - (\lambda_0 I - A_\varepsilon)^{-1}]\chi_{\mathcal{O}}.$$

Convergence rate of A towards A_ε

- $v = (\lambda_0 I - A)^{-1} Pf$ is solution of

$$\lambda_0 v - \nu \Delta v + (v \cdot \nabla) v_s + (v_s \cdot \nabla) v + \nabla q = f \ \text{ in } \Omega,$$
$$\operatorname{div} v = 0 \text{ in } \Omega, \quad v = 0 \ \text{ on } \Gamma.$$

- $v^\varepsilon = (\lambda_0 I - A_{v_s^\varepsilon})^{-1} Pf$ is solution of

$$\lambda_0 v - \nu \Delta v + (v \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) v + \nabla q = f \ \text{ in } \Omega,$$
$$\operatorname{div} v = 0 \text{ in } \Omega, \quad v = 0 \ \text{ on } \Gamma.$$

- $v_\varepsilon = (\lambda_0 I - A_\varepsilon)^{-1}$ is solution of

$$\lambda_0 v_\varepsilon - \nu \Delta v_\varepsilon + (v_\varepsilon \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) v_\varepsilon + \nabla q_\varepsilon = f \ \text{ in } \Omega,$$
$$\operatorname{div} v_\varepsilon + \varepsilon q_\varepsilon = 0 \text{ in } \Omega, \quad v_\varepsilon = 0 \ \text{ on } \Gamma.$$

Convergence rate of A towards A_ε

The difference $z_\varepsilon = v_\varepsilon - v^\varepsilon$ obeys

$$\lambda_0 z_\varepsilon - \nu\Delta z_\varepsilon + (z_\varepsilon \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) z_\varepsilon + \nabla p_\varepsilon = 0 \ \text{ in } \Omega,$$
$$\operatorname{div} z_\varepsilon + \varepsilon p_\varepsilon = -\varepsilon q \text{ in } \Omega, \quad z_\varepsilon = 0 \ \text{ on } \Gamma.$$

With the adjoint system

$$\lambda_0 \Phi_\varepsilon - \nu\Delta\Phi_\varepsilon + (\nabla v_s^\varepsilon)^T \Phi_\varepsilon - (v_s^\varepsilon \cdot \nabla)\Phi_\varepsilon + \nabla\psi_\varepsilon - \operatorname{div}(v_s^\varepsilon)\Phi_\varepsilon$$
$$= v_\varepsilon - v^\varepsilon \ \text{ in } \Omega,$$
$$\operatorname{div} \Phi_\varepsilon + \varepsilon\psi_\varepsilon = 0 \text{ in } \Omega, \quad \Phi_\varepsilon = 0 \ \text{ on } \Gamma,$$

we obtain

$$\int_\Omega |v_\varepsilon - v^\varepsilon|^2 \mathrm{d}x = \varepsilon \int_\Omega q\psi_\varepsilon \mathrm{d}x$$
$$\leq \varepsilon \|q\|_{L^2(\Omega)} \|\psi_\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon \|v_\varepsilon - v^\varepsilon\|_{L^2(\Omega)}.$$

Detectability of $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$

- We want to construct feedbacks using Riccati equations associated with $(A, B, \mathcal{C}|_Z)$ and $(A_\varepsilon, B_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$
- We assume that $\mathcal{C} \in \mathcal{L}(H, Y)$, Y is a Hilbert space, and that

$$(A, \mathcal{C}|_Z) \quad \text{is detectable in } Z = V_n^0(\Omega).$$

- We prove that

$$(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon} = \mathcal{C}) \quad \text{is detectable in } Z_\varepsilon = (L^2(\Omega))^3 = H,$$

uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, with the same arguments as those used to deduce the stabilizatility of $(A_\varepsilon, B_\varepsilon)$ from that of (A, B).

The Riccati equations

We introduce Π

$$\Pi \in \mathcal{L}(Z), \quad \Pi = \Pi^* \geq 0, \quad B^*\Pi \in \mathcal{L}(Z, U),$$
$$\Pi A + A^*\Pi - \Pi BB^*\Pi + P^*\mathcal{C}^*\mathcal{C}P = 0.$$

The algebraic Riccati equation of the approximate system is

$$\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon), \quad \Pi_\varepsilon = \Pi_\varepsilon^* \geq 0, \quad B_\varepsilon^*\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon, U),$$
$$\Pi_\varepsilon A_\varepsilon + A_\varepsilon^*\Pi_\varepsilon - \Pi_\varepsilon B_\varepsilon B_\varepsilon^*\Pi_\varepsilon + \mathcal{C}^*\mathcal{C} = 0.$$

Convergence rate for the feedback gains

There exist $\omega_\Pi^* > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|e^{(A_\varepsilon - B_\varepsilon \Pi_\varepsilon)t}\|_{\mathcal{L}(Z_\varepsilon)} \leq Ce^{-\omega_\Pi^* t}, \quad \forall t \geq 0.$$

We have

$$\|\Pi P - \Pi_\varepsilon\|_{\mathcal{L}(H)} \leq C\varepsilon\,|\ln(\varepsilon)|, \quad \forall \varepsilon \in (0,\varepsilon_0),$$

and

$$\|B^*\Pi P - B_\varepsilon^*\Pi_\varepsilon\|_{\mathcal{L}(H,U)} \leq C\varepsilon|\ln\varepsilon|, \quad \forall \varepsilon \in (0,\varepsilon_0).$$

Convergence rates for the closed-loop systems

- $y(t) = e^{(A+BK)t} y_0, \quad K = -\Pi.$

  $\|e^{(A+BK)t}\|_{\mathcal{L}(Z)} \leq C e^{-\omega_\Pi t}, \quad \forall t \geq 0.$
- $y^\varepsilon(t) = e^{(A+BK_\varepsilon)t} y_0, \quad K_\varepsilon = -\Pi_\varepsilon.$
- $y_\varepsilon(t) = e^{(A_\varepsilon+B_\varepsilon K_\varepsilon)t} y_0, \quad K_\varepsilon = -\Pi_\varepsilon.$

For all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|y(t) - y_\varepsilon(t)\|_H \leq C \, \frac{e^{(-\omega_\Pi + \varrho\varepsilon|\ln \varepsilon|)t}}{t} \, \varepsilon|\ln \varepsilon| \|y_0\|_H,$$

$$\|y_\varepsilon - y\|_{L^p(0,\infty;H)} \leq C_p \, \varepsilon^{1/p} |\ln \varepsilon|^{1/p} \|y_0\|_H, \quad \forall p \in (1, \infty),$$

$$\|y_\varepsilon - y^\varepsilon\|_{L^p(0,\infty;H)} \leq C_p \, \varepsilon^{1/p} \|y_0\|_H, \quad \forall p \in (1, \infty).$$

$$\frac{\partial y}{\partial t} + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s - \nu \, \Delta y + \nabla q = 0 \quad \text{in} \quad Q,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad Q, \quad y(x,t) = \sum_{i=1}^{N_c} u_i(t) \, g_i(x) \quad \text{on} \quad \Sigma,$$

$$y(0) = y_0 \quad \text{in} \quad \Omega,$$

with the control space $U = \mathbb{R}^{N_c}$.

The Oseen system is a differential system for Py ($Py' = APy + Bu$) coupled with an algebraic equation for $(I - P)y$.

Main goal. Determine $K = (K_1, \cdots, K_{N_c}) \in \mathcal{L}(Z, U)$ such $(e^{t(A+\omega I+BK)})_{t \geq 0}$ is exponentially stable with $\omega > 0$.

Assumption. $g_i \in H^{3/2}(\Gamma)$, $\int_\Gamma g_i \cdot n \, dx = 0$. $(A + \omega_u I, B)$ is exponentially stabilizable, with $\omega_u > \omega > 0$.

Approximation by the pseudo-compressible model

$$\frac{\partial y_\varepsilon}{\partial t} - \nu \Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) y_\varepsilon + \nabla p_\varepsilon = 0 \text{ in } Q,$$
$$\operatorname{div} y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q,$$
$$y_\varepsilon = \sum_{i=1}^{N_c} u_i(t)\, g_i \text{ on } \Sigma,$$
$$y_\varepsilon(0) = y_0 \text{ in } \Omega.$$

- $v = Dg$ is solution of

$$\lambda_0 v - \nu \Delta v + (v \cdot \nabla) v_s + (v_s \cdot \nabla) v + \nabla q = 0 \text{ in } \Omega,$$
$$\operatorname{div} v = 0 \text{ in } \Omega, \quad v = g \text{ on } \Gamma.$$

- $v_\varepsilon = D_\varepsilon g$ is solution of

$$\lambda_0 v_\varepsilon - \nu \Delta v_\varepsilon + (v_\varepsilon \cdot \nabla) v_s^\varepsilon + (v_s^\varepsilon \cdot \nabla) v_\varepsilon + \nabla q_\varepsilon = 0 \text{ in } \Omega,$$
$$\operatorname{div} v_\varepsilon + \varepsilon q_\varepsilon = 0 \text{ in } \Omega, \quad v_\varepsilon = g \text{ on } \Gamma.$$

Difficulty

The Oseen system is a differential algebraic system of the form

$$Py'(t) = APy(t) + Bu, \quad B = (\lambda_0 I - A) \sum_{i=1}^{N_c} u_i \, P \, Dg_i,$$
$$(I - P)y(t) = (I - P) \sum_{i=1}^{N_c} u_i(t) \, Dg_i,$$

while the pseudo-compressible Oseen system is of the form

$$y_\varepsilon'(t) = A_\varepsilon y_\varepsilon(t) + B_\varepsilon u, \quad B_\varepsilon = (\lambda_0 I - A_\varepsilon) \sum_{i=1}^{N_c} u_i \, D_\varepsilon g_i.$$

We have good approximation properties for $A - A_\varepsilon$, for $D - D_\varepsilon$

$$\|Dg - D_\varepsilon g\|_{(L^2(\Omega))^3} \leq C \, \varepsilon \, \|g\|_{(H^{1/2}(\Gamma))^3}$$

but not for $P - I$, and thus not for $B - B_\varepsilon$.

Possible remedies

- Change the boundary control operator in the pseudo-compressible Oseen system

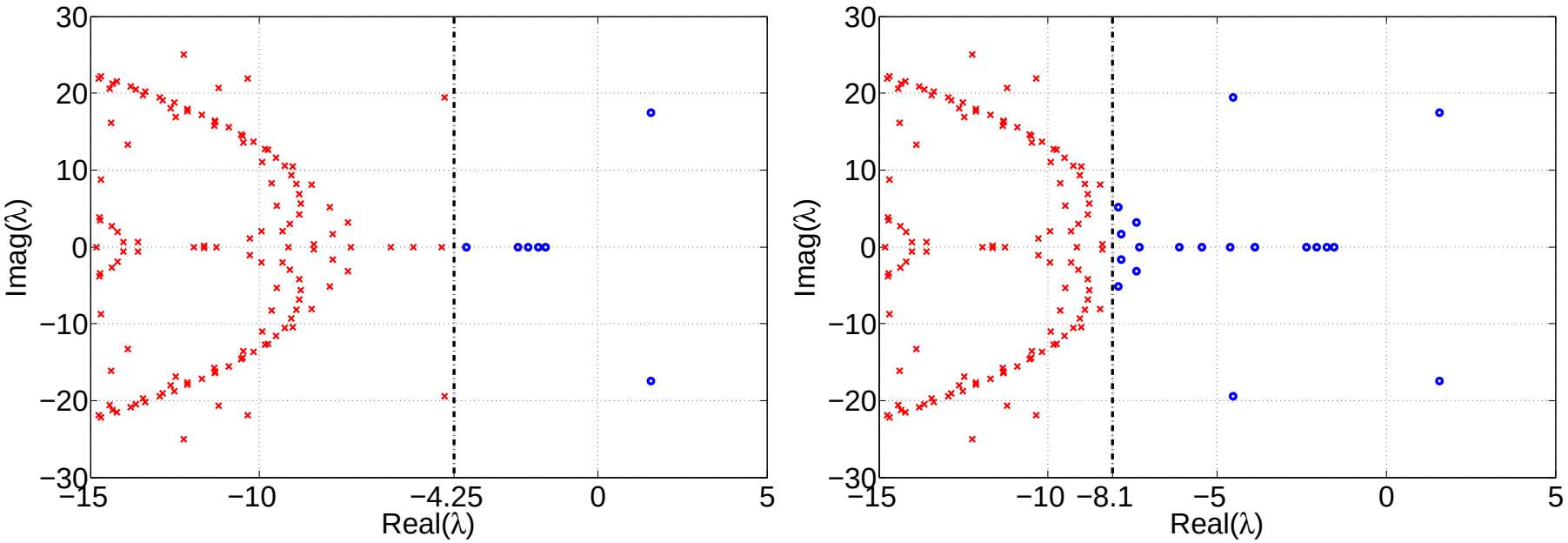
$$y_\varepsilon'(t) = A_\varepsilon y_\varepsilon(t) + B_\varepsilon u, \quad B_\varepsilon = (-A_\varepsilon) \sum_{i=1}^{N_c} u_i \, P_\varepsilon \, D_\varepsilon g_i,$$

where P_ε is an approximation of P.

- Compute a feedback for a ROM based on a spectral projection.

Spectrum of A (and of A_ε)

The resolvent of A (resp. A_ε) is compact in Z (resp. Z_ε).



$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j), \quad Z = Z_u \oplus Z_s, \quad \dim Z_u = d_u < \infty.$$

Z_u and Z_s are invariant subspaces of A.

$$\operatorname{Re}\sigma(A|_{Z_u}) > -\omega_u \quad \text{and} \quad \operatorname{Re}\sigma(A|_{Z_s}) < -\omega.$$

Spectral projections in Z

The projector $P_u \in \mathcal{L}(Z, Z_u)$ (and $P_u \in \mathcal{L}(H, Z_u)$) is defined by

$$P_u = \frac{1}{2i\pi} \int_{\Gamma_u} (\lambda I - A)^{-1} P \, d\lambda,$$

Γ_u is a union of Jordan curves, around $(\lambda_j)_{j \in J_u} \cup (\overline{\lambda_j})_{j \in J_u}$.

We split $z' = Az + Bu$ into two systems

$$A_u = A|_{Z_u}, \qquad A_s = A|_{Z_s}, \quad B_u = P_u B, \quad B_s = (I - P_u) B.$$

$$z = z_u + z_s, \quad z_u' = A_u z_u + B_u u, \quad z_s' = A_s z_s + B_s u.$$

Spectral projections in Z_ε - Approximation of P_u

There exist $\varepsilon_0 > 0$, such that $\Gamma_u \subset \rho(A_\varepsilon),\ \forall \varepsilon \in (0, \varepsilon_0)$.

We set

$$P_{\varepsilon,u} = \frac{1}{2i\pi} \int_{\Gamma_u} (\lambda I - A_\varepsilon)^{-1} P_\varepsilon \, d\lambda,$$

$$Z_{\varepsilon,u} = P_{\varepsilon,u} Z_\varepsilon \quad \text{and} \quad Z_{\varepsilon,s} = (I - P_{\varepsilon,u}) Z_\varepsilon,$$

$$A_{\varepsilon,u} = A_\varepsilon\big|_{Z_{\varepsilon,u}}, \quad A_{\varepsilon,s} = A_\varepsilon\big|_{Z_{\varepsilon,s}},$$

$$B_{\varepsilon,u} = P_{\varepsilon,u} B_\varepsilon \quad B_{\varepsilon,s} = (I - P_{\varepsilon,u}) B_\varepsilon.$$

We split $z'_\varepsilon = A_\varepsilon z_\varepsilon + B_\varepsilon u$ into two systems

$$z_\varepsilon = z_{\varepsilon,u} + z_{\varepsilon,s}, \quad z'_{\varepsilon,u} = A_{\varepsilon,u} z_{\varepsilon,u} + B_{\varepsilon,u} u, \quad z'_{\varepsilon,s} = A_{\varepsilon,s} z_{\varepsilon,s} + B_{\varepsilon,s} u.$$

We choose $\varepsilon_0 > 0$, and $\exists C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$:

$$\|P_u - P_{\varepsilon,u}\|_{\mathcal{L}(H)} \leq C\,\varepsilon \quad \text{and} \quad \dim(Z_u) = \dim(Z_{\varepsilon,u}).$$

Estimates in Z_u and $Z_{\varepsilon,u}$

- $B_u u = \sum_{i=1}^{N_c} u_i(\lambda_0 I - A)P_u D\, g_i$,
- $B_{\varepsilon,u} u = \sum_{i=1}^{N_c} u_i(\lambda_0 I - A_\varepsilon)P_{\varepsilon,u} D_\varepsilon\, g_i$.

$$\|(\lambda_0 I - A_u)^{-1}P_u - (\lambda_0 I - A_{\varepsilon,u})^{-1}P_{\varepsilon,u}\|_{\mathcal{L}(H)} \leq C\,\varepsilon,$$

$$\|(\lambda_0 I - A_u)^{-1}B_u - (\lambda_0 I - A_{\varepsilon,u})^{-1}B_{\varepsilon,u}\|_{\mathcal{L}(U,H)} \leq C\,\varepsilon,$$

$$\|B_u - B_{\varepsilon,u}\|_{\mathcal{L}(U,H)} \leq C\,\varepsilon, \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

Uniform bound for $B_{\varepsilon,u} \in \mathcal{L}(U,H)$

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \|B_{\varepsilon,u}\|_{\mathcal{L}(U,H)} < +\infty.$$

Uniform stabilizability of $(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}, B_{\varepsilon,u})$

Assumption: Stabilizability and detectability conditions.

$(A_u + \omega_u P_u, B_u)$ is stabilizable in Z_u.

Either $(A_u + \omega_u P_u, \mathcal{C}|_{Z_u})$ is detectable or $\mathcal{C} = 0$.

We choose $\varepsilon_0 > 0$ such that

$$\operatorname{Re}\sigma(A_{\varepsilon,u}) > -\omega_u \quad \text{and} \quad \operatorname{Re}\sigma(A_{\varepsilon,s}) < -\omega, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

$(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}, B_{\varepsilon,u})$ is unif. stabilizable

$(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}, \mathcal{C}\big|_{Z_{\varepsilon,u}})$ is unif. detectable.

The Riccati equation in Z_u

$$\Pi_u \in \mathcal{L}(Z_u, Z_u^*), \quad \Pi_u = \Pi_u^* \geq 0, \quad C_u = \mathcal{C}|_{Z_u},$$

$$\Pi_u(A_u + \omega_u P_u) + (A_u^* + \omega_u P_u^*)\Pi_u - \Pi_u B_u B_u^* \Pi_u + C_u^* C_u = 0,$$

$A_u + \omega_u P_u - B_u B_u^* \Pi_u$ is exponentially stable in Z_u.

If $K_u = -B_u^* \Pi_u$, then

$$\|e^{t(A+BK_u)}\|_{\mathcal{L}(H)} \leq Ce^{-t\omega}.$$

The Riccati equation in $Z_{\varepsilon,u}$ and convergence rates

$$\Pi_{\varepsilon,u} \in \mathcal{L}(Z_{\varepsilon,u}, Z^*_{\varepsilon,u}), \quad \Pi_{\varepsilon,u} = P^*_{\varepsilon,u}, \quad C_{\varepsilon,u} = \mathcal{C}\big|_{Z_{\varepsilon,u}},$$

$$\Pi_{\varepsilon,u}(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}) + (A^*_{\varepsilon,u} + \omega_u P^*_{\varepsilon,u})\Pi_{\varepsilon,u} - \Pi_{\varepsilon,u} B_{\varepsilon,u} B^*_{\varepsilon,u} \Pi_{\varepsilon,u} + C^*_{\varepsilon,u} C_{\varepsilon,u} = 0,$$

$A_{\varepsilon,u} + \omega_u P_{\varepsilon,u} - B_{\varepsilon,u} B^*_{\varepsilon,u} \Pi_{\varepsilon,u}$ is exponentially stable in $Z_{\varepsilon,u}$.

The solutions Π_u and $\Pi_{\varepsilon,u}$, and the feedbacks $K_u = -B^*_u \Pi_u P_u$ and $K_{\varepsilon,u} = -B^*_{\varepsilon,u} \Pi_{\varepsilon,u} P_{\varepsilon,u}$ obey

$$\|\Pi_u P_u - \Pi_{\varepsilon,u} P_{\varepsilon,u}\|_{\mathcal{L}(H)} \leq C\,\varepsilon,$$

and

$$\|K_u - K_{\varepsilon,u}\|_{\mathcal{L}(H,U)} \leq C\,\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Convergence rates for the closed-loop systems

- $Py(t) = e^{(A+BK)t}y_0, \quad K = -B_u^*\Pi_u P_u$, u is the control.

  $(I-P)y(t) = \sum_{i=1}^{N_c} K_i y(t)(I-P)Dg_i$.
- $Py^\varepsilon(t) = e^{(A+BK_\varepsilon)t}y_0, \quad K_\varepsilon = -B_{\varepsilon,u}^*\Pi_{\varepsilon,u}P_{\varepsilon,u}$, u^ε is the control.

  $(I-P)y^\varepsilon(t) = \sum_{i=1}^{N_c} K_{\varepsilon,i} y^\varepsilon(t)(I-P)Dg_i$.
- $y_\varepsilon(t) = e^{(A_\varepsilon+B_\varepsilon K_\varepsilon)t}y_0, \quad K_\varepsilon = -B_{\varepsilon,u}^*\Pi_{\varepsilon,u}P_{\varepsilon,u}$, u_ε is the control.

For all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|u_\varepsilon(t) - u(t)\|_U \le C\,\frac{e^{(-\omega+\varrho\varepsilon)t}}{t}\,\varepsilon\|y_0\|_H,$$

$$\|u_\varepsilon - u\|_{L^p(0,\infty;U)} \le C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

$$\|u_\varepsilon - u^\varepsilon\|_{L^p(0,\infty;U)} \le C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

where $\omega > 0$ is the a priori prescribed decay rate.

Convergence rates for the closed-loop systems

• Convergence rates for the projections of the solutions of closed-loop systems

$$\|P_{\varepsilon,u}y_\varepsilon(t) - P_u y(t)\|_H \leq C\,\frac{e^{(-\omega+\varrho\varepsilon)t}}{t}\,\varepsilon\|y_0\|_H,$$

$$\|P_{\varepsilon,u}y_\varepsilon - P_u y\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

$$\|P_{\varepsilon,u}y_\varepsilon - P_u y^\varepsilon\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty).$$

• We could obtain convergence rates between y_ε and y, on compact time intervals $[0, T]$, if we took a dynamic controller and if $y_0 \in V_n^0(\Omega) \cap (H_0^1(\Omega))^3$.

HAPPY BIRTHDAY MARIUS