Homogenization for Problems with Sign-Changing Coefficients

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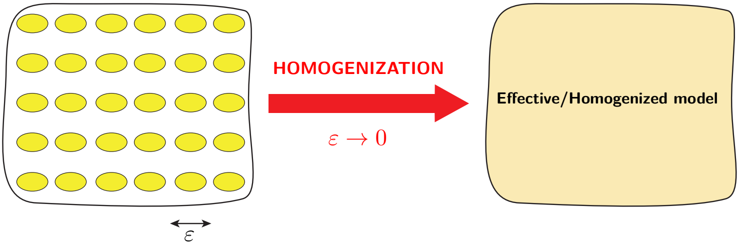
Periodic Homogenization



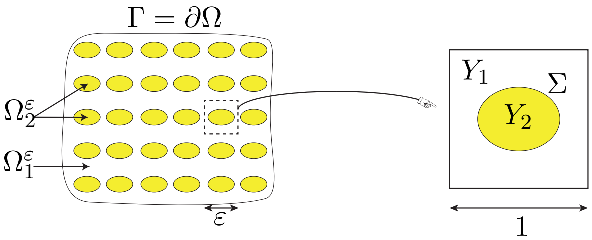
Goal

Derive, via an asymptotic analysis, **simpler macroscopic models** from **complicated microscopic models**, especially for numerical simulations.

Applications

Photonic crystals, composite materials, porous media,...

Periodic Homogenization



- $y \in Y = (0,1)^2$: **microscopic (fast) variable**

$$1\text{–periodic conductivity : } a(y) = \begin{cases} a_1 & \text{for } y \in Y_1 \\ a_2 & \text{for } y \in Y_2 \end{cases}$$

- $x \in \Omega \subset \mathbb{R}^2$: **macroscopic (slow) variable**

$$\varepsilon\text{–periodic conductivity : } a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = \begin{cases} a_1 & \text{for } x \in \Omega_1^\varepsilon \\ a_2 & \text{for } x \in \Omega_2^\varepsilon \end{cases}$$

Periodic Homogenization

$$(\mathcal{P}^\varepsilon)\left\{\begin{array}{rcll} -\mathrm{div}\,(a^\varepsilon(x)\nabla u^\varepsilon) & = & f, & \text{in } \Omega \\ u^\varepsilon & = & 0, & \text{on } \Gamma. \end{array}\right.$$

Variational formulation: Find $u^\varepsilon \in H_0^1(\Omega)$ such that:

$$\int_\Omega a^\varepsilon(x)\nabla u^\varepsilon \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x, \qquad \forall v \in H_0^1(\Omega).$$

1. Does u^ε have a limit ?
2. If so, what problem does this limit solve ?

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For **elliptic problems** ($i.e.$ when $a_1 > 0$ and $a_2 > 0$), these questions are very well understood.

Periodic Homogenization

The **homogenized problem** reads

$$(\mathcal{P}) \begin{cases} -\mathrm{div}\left(a^H \nabla u\right) & = & f, & \text{in } \Omega \\ u & = & 0, & \text{on } \Gamma, \end{cases}$$

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where

- $a^H = (a^H_{ij})_{1 \leqslant i,j \leqslant 2}$ is the **constant symmetric matrix**:

$$a^H_{ij} = \int_Y a(y)(\nabla \chi_i + e_i) \cdot (\nabla \chi_j + e_j)\, \mathrm{d}y,$$

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- $\chi_1, \chi_2 \in H^1_\#(Y)/\mathbb{R}$ solve the **cell problems**:

$$\begin{cases} -\operatorname{div}\left(a(y)\nabla\chi_i\right) = \operatorname{div}\left(a(y)e_i\right), \quad \text{in } Y \\ \chi_i \text{ is } Y\text{-periodic.} \end{cases}$$

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Variational formulation: Find $u \in H^1_0(\Omega)$ such that:

$$\int_\Omega a^H(x)\nabla u \cdot \nabla v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x, \qquad \forall v \in H^1_0(\Omega).$$

Theorem

In the elliptic case, problems $(\mathcal{P}^\varepsilon)$ *and* $(\mathcal{P})$ *are well-posed in* $H_0^1(\Omega)$ *and the sequence* (u^ε) *weakly converges to* u *in* $H_0^1(\Omega)$*.*

The proof is based on three steps:

$\boxed{1}$ $(\mathcal{P}^\varepsilon)$ **is well-posed and we have:** $\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C$.

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$\boxed{\mathbf{3}}$ $(\mathcal{P})$ **is well-posed** as the matrix a^H is **positive definite**:

$$a^H \xi \cdot \xi \geqslant \left(\int_Y a^{-1}(y)\, \mathrm{d}y \right)^{-1} |\xi|^2 \qquad \forall \xi = (\xi_1, \xi_2)^{\mathrm{T}} \in \mathbb{R}^2.$$

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OUR GOAL

What happens for **non elliptic** problems ($a_1 > 0$ and $a_2 < 0$) ?

$$\int_\Omega a^\varepsilon \nabla u \cdot \nabla v,\ \int_\Omega a^H \nabla u \cdot \nabla v \text{ are not coercive } \Longrightarrow \quad \cancel{\boxed{\mathbf{1}}} \quad \cancel{\boxed{\mathbf{3}}}$$

METAMATERIALS

Outline

Outline

Main Result

Theorem

Assume that $a_1 > 0$ *and* $a_2 < 0$ *and define the contrast*

$$\kappa := \frac{a_1}{|a_2|}.$$

Then, there exists two constants $\kappa_Y, \kappa'_Y > 0$ *such that for*

$$\kappa > \kappa_Y \qquad \textit{or} \qquad \kappa < 1/\kappa'_Y,$$

problems $(\mathcal{P}^\varepsilon)$ *and* $(\mathcal{P})$ *are well-posed and* u^ε *weakly converges to* u *in* $H^1_0(\Omega)$.

Comments

- κ_Y **and** κ'_Y
  Depend only on the geometry of the reference cell, and are given by continuity constants of some harmonic extension operators (from Y_1 to Y_2 or vice versa).

- **Bibliography**
  - First proved by Bunoiu-R. (2016) for large contrasts using the T-coercivity method...
  - ...generalized by Bonnetier, Dapogny and Triki (2019) to small contrasts using the Neumann–Poincaré Operator...
  - ...and to other scalar sign-changing problems (Dirichlet and Neumann) and Maxwell's system by Bunoiu-Chesnel-R.-Rihani (2021).

Outline

From coercivity to $\mathtt{T}$-coercivity

$$\text{Find } u \in V, \mathcal{A}(u,v) = \langle f, v \rangle, \quad \forall v \in V.$$

Coercivity : There exists $\gamma > 0$ such that

$$\mathcal{A}(u,u) \geqslant \gamma \|u\|^2, \qquad \forall\, u \in V.$$

From coercivity to $\mathtt{T}$-coercivity

$$\text{Find } u \in V, \mathcal{A}(u,v) = \langle f, v\rangle, \quad \forall v \in V$$

$\mathtt{T}$-Coercivity: There exists $\gamma > 0$ and $\mathbf{T} \in \mathcal{L}(V)$ invertible such that

$$\mathcal{A}(u, \mathbf{T}u) \geqslant \gamma \|u\|^2, \qquad \forall\, u \in V.$$

Theorem

Let $\mathcal{A}^\varepsilon(\cdot,\cdot)$ be a uniformly continuous bilinear form on a Hilbert space V such that there exists a family $(\mathbf{T}^\varepsilon)$ of uniformly boundedly invertible operators on V satisfying

$$\exists \gamma > 0 : \quad \mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) \geqslant \gamma \|u\|^2, \qquad \forall\, u \in V.$$

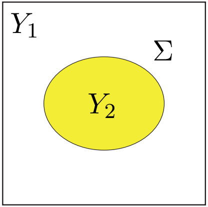
Then the variational problem

$$\textit{Find } u^\varepsilon \in V, \mathcal{A}^\varepsilon(u^\varepsilon, v) = \langle f, v \rangle, \quad \forall v \in V,$$

is well-posed and we have the uniform estimate: $\|u^\varepsilon\| \leqslant C\|f\|$.

- Introduced by Bonnet-Ben Dhia, Ciarlet Jr. *et al.* (2008, 2012,...) to study non elliptic problems (well-posedness, numerical analysis) and Helmholtz type problems.
- For symmetric forms, the invertibility of $(\mathbf{T}^\varepsilon)$ can be dropped and $\mathtt{T}$-coercivity is equivalent to the inf$-$sup condition.

Outline



The weak formulation of the cell problems in the space

$$V_{\#} := \left\{ u \in H^1_{\#}(Y) \mid \int_Y u = 0 \right\}, \qquad \|u\|_{V_{\#}} := \|\nabla u\|_{L^2(Y)}$$

takes the form

$$\text{Find } u \in V_{\#} \text{ such that } \forall v \in V_{\#} :$$

$$\mathcal{A}(u,v) := \int_Y a(y) \nabla u \cdot \nabla v = \int_Y f v.$$

For all $u \in V_\#$, setting $u_1 := u_{|Y_1}$, $u_2 := u_{|Y_2}$:

$$\begin{aligned}\mathcal{A}(u, \mathbf{T}u) &= \int_Y a(y) \nabla u \cdot \nabla(\mathbf{T}u) \\ &= a_1 \int_{Y_1} \nabla u_1 \cdot \nabla(\mathbf{T}u) + a_2 \int_{Y_2} \nabla u_2 \cdot \nabla(\mathbf{T}u)\end{aligned}$$

$$\mathbf{T}u = \begin{cases} \dots & \text{in } Y_1 \\ \dots & \text{in } Y_2. \end{cases}$$

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$$\mathbf{T}u = \begin{cases} u_1 & \text{in } Y_1 \\ -u_2 & \text{in } Y_2. \end{cases}$$

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$$\mathbf{T}u = \begin{cases} u_1 & \text{in } Y_1 \\ -u_2 + 2\mathbf{P}(u_1) & \text{in } Y_2. \end{cases}$$

where $\mathbf{P}$ denotes the **harmonic extension** from Y_1 to Y_2.

For all $u \in V_\#$, setting $u_1 := u_{|Y_1}$, $u_2 := u_{|Y_2}$:

$$\begin{aligned} \mathcal{A}(u, \mathbf{T}u) &= \int_Y a(y)\nabla u \cdot \nabla(\mathbf{T}u) \\ &= a_1 \int_{Y_1} \nabla u_1 \cdot \nabla(\mathbf{T}u) + a_2 \int_{Y_2} \nabla u_2 \cdot \nabla(\mathbf{T}u) \end{aligned}$$

$$\widetilde{\mathbf{T}}u = \begin{cases} u_1 & \text{in } Y_1 \\ -u_2 + 2\mathbf{P}(u_1) & \text{in } Y_2. \end{cases} \qquad \mathbf{T}u := \widetilde{\mathbf{T}}u - \left(\int_Y \widetilde{\mathbf{T}}u\right) \in V_\#.$$

We cleary have:

$$\mathbf{T} \in \mathcal{L}(V_\#).$$

It is well-known[1] that there exists $\kappa_Y > 0$ such that:

$$\|\nabla(\mathbf{P}u_1)\|^2_{L^2(Y)} \leqslant \kappa_Y \|\nabla u_1\|^2_{L^2(Y_1)}, \qquad \forall u \in H^1(Y_1).$$

Proposition

For $\kappa > \kappa_Y$, there exists $\gamma > 0$ such that for all $u \in V_\#$:

$$\mathcal{A}(u, \mathbf{T}u) = \int_Y a(y)\nabla u(y) \cdot \nabla(\mathbf{T}u)(y)\ \mathrm{d}y \geqslant \gamma \|\nabla u\|^2_{L^2(Y)}.$$

[1] See e.g. Lemma 2.9 in the book of Cioranescu and Saint Jean Paulin

Proof (Large Contrasts)

$$\mathbf{T}u := \widetilde{\mathbf{T}}u - \left(\int_Y \widetilde{\mathbf{T}}u\right), \qquad \widetilde{\mathbf{T}}u = \begin{cases} u_1 & \text{in } Y_1 \\ -u_2 + 2\mathbf{P}(u_1) & \text{in } Y_2. \end{cases}$$

$$\mathcal{A}(u, \mathbf{T}u) \;\; = \;\; \int_Y a(y)\nabla u(y) \cdot \nabla(\widetilde{\mathbf{T}}u)(y)\, \mathrm{d}y$$

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$$\begin{aligned} \mathcal{A}(u, \mathbf{T}u) &= \int_Y a(y)\nabla u(y) \cdot \nabla(\widetilde{\mathbf{T}}u)(y)\, \mathrm{d}y \\ &= a_1 \int_{Y_1} |\nabla u_1|^2 + |a_2| \int_{Y_2} |\nabla u_2|^2 + 2a_2 \int_{Y_2} \nabla u_2 \cdot \nabla(\mathbf{P}u_1) \end{aligned}$$

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$$\begin{aligned}
\mathcal{A}(u, \mathbf{T}u) &= \int_Y a(y)\nabla u(y) \cdot \nabla(\widetilde{\mathbf{T}}u)(y)\,\mathrm{d}y \\
&= a_1 \int_{Y_1} |\nabla u_1|^2 + |a_2| \int_{Y_2} |\nabla u_2|^2 + 2a_2 \int_{Y_2} \nabla u_2 \cdot \nabla(\mathbf{P}u_1) \\
&\geqslant \kappa |a_2| \int_{Y_1} |\nabla u_1|^2 + |a_2| \int_{Y_2} |\nabla u_2|^2 \\
&\quad - |a_2|\eta \int_{Y_2} |\nabla u_2|^2 - \frac{|a_2|}{\eta} \int_{Y_2} |\nabla(\mathbf{P}u_1)|^2 \qquad \text{(Young)}
\end{aligned}$$

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\mathcal{A}(u, \mathbf{T}u) &= \int_Y a(y)\nabla u(y) \cdot \nabla(\widetilde{\mathbf{T}}u)(y)\ \mathrm{d}y \\
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&\geqslant \kappa|a_2| \int_{Y_1} |\nabla u_1|^2 + |a_2| \int_{Y_2} |\nabla u_2|^2 \\
&\quad -|a_2|\eta \int_{Y_2} |\nabla u_2|^2 - \frac{|a_2|}{\eta} \int_{Y_2} |\nabla(\mathbf{P}u_1)|^2 \qquad \text{(Young)} \\
&\geqslant |a_2| \left\{ \left(\kappa - \frac{\kappa_Y}{\eta}\right) \int_{Y_1} |\nabla u_1|^2 + (1-\eta) \int_{Y_2} |\nabla u_2|^2 \right\}
\end{aligned}$$

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&\geqslant \gamma \int_Y |\nabla u|^2, \qquad \text{if } \kappa > \kappa_Y \text{ and } \eta \in (\kappa_Y/\kappa, 1).
\end{aligned}$$

■

To handle the case of **small contrasts**, we introduce the Dirichlet harmonic extension operator from Y_2 to Y_1:

$$\begin{cases} -\Delta(\mathbf{Q}\varphi) &= 0 & \text{in } Y_1, \\ \mathbf{Q}\varphi &= \varphi & \text{on } \partial Y_2, \\ \mathbf{Q}\varphi &= 0 & \text{on } \partial Y. \end{cases}$$

It can be proved[2] that there exists $\kappa'_Y > 0$ such that

$$\|\nabla(\mathbf{Q}\varphi)\|^2_{L^2(Y)} \leqslant \kappa'_Y \|\nabla\varphi\|^2_{L^2(Y_2)}, \qquad \forall \varphi \in H^1_{\mathrm{mean}}(Y_2),$$

where

$$H^1_{\mathrm{mean}}(Y_2) = \left\{\varphi \in H^1(Y_2) \mid \mathcal{M}_2(\varphi) = 0\right\}, \quad \mathcal{M}_2(\varphi) = \frac{1}{|Y_2|}\int_{Y_2} \varphi \,\mathrm{d}y.$$

[2] See, e.g., Lemma 2.3. in Cazeaux-Grandmont-Maday, 2015.

Define $\mathbf{T} \in \mathcal{L}(V_\#)$ by

$$\mathbf{T}u := \widetilde{\mathbf{T}}u - \left(\int_Y \widetilde{\mathbf{T}}u \right),$$

where

$$\widetilde{\mathbf{T}}u = \begin{cases} u_1 - 2\mathbf{Q}(u_2 - \mathcal{M}_2(u_2)) & \text{in } Y_1 \\ -u_2 + 2\mathcal{M}_2(u_2) & \text{in } Y_2, \end{cases}$$

Proposition

For $\kappa < 1/\kappa'_Y$, there exists $\gamma' > 0$ such that for all $u \in V_\#$:

$$\mathcal{A}(u, \mathbf{T}u) = \int_Y a(y)\nabla u(y) \cdot \nabla(\mathbf{T}u)(y)\,\mathrm{d}y \geqslant \gamma' \|\nabla u\|^2_{L^2(Y)}.$$

Theorem

Assume that

$$\kappa > \kappa_Y \qquad \text{or} \qquad \kappa < 1/\kappa'_Y$$

Then, the cell problems, $i = 1, 2$ *:*

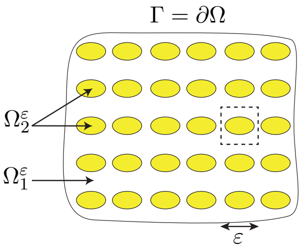
$$\begin{cases} -\mathrm{div}\,(a(y)\nabla\chi_i) = \mathrm{div}\,(a(y)e_i)\,, & \text{in } Y \\ \chi_i \text{ is } Y\text{-periodic}, \end{cases}$$

are well-posed.

Outline

Large Contrasts

$$\mathcal{A}^\varepsilon(u,v) = \int_\Omega a^\varepsilon(x)\nabla u(x)\nabla v(x)dx$$



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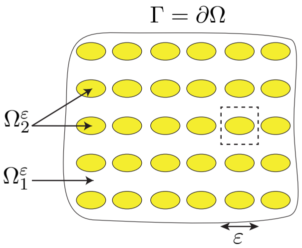

$$\forall u \in H_0^1(\Omega): \quad \mathbf{T}^\varepsilon u = \begin{cases} u_1 & \text{in } \Omega_1^\varepsilon \\ -u_2 + 2\mathbf{P}^\varepsilon(u_1) & \text{in } \Omega_2^\varepsilon. \end{cases}$$

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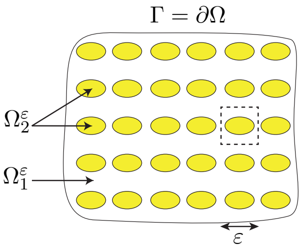

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$$\forall x = \varepsilon(y+k) \in \Omega \ : \ (\mathbf{P}^\varepsilon u)(x) := (\mathbf{P}u_k^\varepsilon)(y), \quad u_k^\varepsilon(y) := u(x).$$

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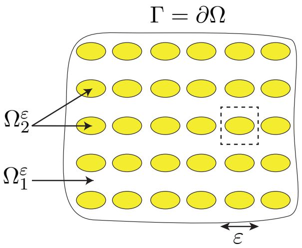

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For all $u \in H_0^1(\Omega)$, we have: $\displaystyle\int_{\Omega_2^\varepsilon} |\nabla(\mathbf{P}^\varepsilon u)|^2 \leqslant \kappa_Y \int_{\Omega_1^\varepsilon} |\nabla u|^2.$

Large Contrasts

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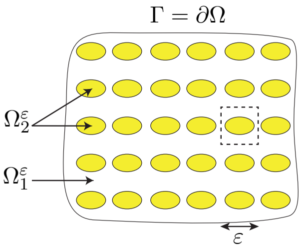

$$\forall u \in H_0^1(\Omega): \quad \mathbf{T}^\varepsilon u = \begin{cases} u_1 & \text{in } \Omega_1^\varepsilon \\ -u_2 + 2\mathbf{P}^\varepsilon(u_1) & \text{in } \Omega_2^\varepsilon. \end{cases}$$

$$\forall x = \varepsilon(y+k) \in \Omega \ : \ (\mathbf{P}^\varepsilon u)(x) := (\mathbf{P}u_k^\varepsilon)(y), \quad u_k^\varepsilon(y) := u(x).$$

For all $u \in H_0^1(\Omega)$, we have: $\displaystyle\int_{\Omega_2^\varepsilon} |\nabla(\mathbf{P}^\varepsilon u)|^2 \leqslant \kappa_Y \int_{\Omega_1^\varepsilon} |\nabla u|^2.$

$$\Longrightarrow \quad \int_\Omega a^\varepsilon(x)\nabla u \cdot \nabla(\mathbf{T}^\varepsilon u) \geqslant \gamma \int_\Omega |\nabla u|^2, \qquad \forall \kappa > \kappa_Y.$$

Small Contrasts

$$\mathcal{M}_2^\varepsilon(u)(x) = \frac{1}{|\varepsilon Y_2^{\mathbf{k}}|} \int_{\varepsilon Y_2^{\mathbf{k}}} u \, \mathrm{d}x, \qquad \forall x \in \varepsilon Y_2^{\mathbf{k}}.$$

$$H^1_{\mathrm{mean}}(\Omega_2^\varepsilon) = \left\{ u \in H^1(\Omega_2^\varepsilon) \mid \mathcal{M}_2^\varepsilon(u) = 0 \right\}.$$

$$\mathbf{T}^\varepsilon u = \begin{cases} u_1 - 2\mathbf{Q}^\varepsilon(u_2 - \mathcal{M}_2^\varepsilon(u_2)) & \text{in } \Omega_1^\varepsilon, \\ -u_2 + 2\mathcal{M}_2^\varepsilon(u_2) & \text{in } \Omega_2^\varepsilon. \end{cases} \tag{1}$$

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We have

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$$\Longrightarrow \quad \int_\Omega a^\varepsilon(x) \nabla u \cdot \nabla(\mathbf{T}^\varepsilon u) \geqslant \gamma' \int_\Omega |\nabla u|^2, \qquad \forall \kappa < 1/\kappa'_Y.$$

Theorem

Assume that

$$\kappa > \kappa_Y \qquad \text{or} \qquad \kappa < 1/\kappa'_Y.$$

Then for every $f \in L^2(\Omega)$*, the problem*

$$(\mathcal{P}^\varepsilon) \begin{cases} -\mathrm{div}\,(a^\varepsilon(x)\nabla u^\varepsilon) &= f, & \text{in } \Omega \\ u^\varepsilon &= 0, & \text{on } \Gamma \end{cases}$$

admits a unique solution u^ε *and there exists* $C > 0$ *such that*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

Outline

One can not pass to the limit directly in $\int_\Omega a^\varepsilon(x)\nabla u^\varepsilon(x)\nabla v(x)dx$, as a^ε and u^ε converge only weakly.

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Two-scale convergence (Nguetseng 89', Allaire 92')

A sequence (u^ε) in $L^2(\Omega)$ *two-scale converges* to a function $u(x,y) \in L^2(\Omega \times Y)$ if for any Y- periodic smooth function $\varphi(x,y)$ defined on $\Omega \times Y$:

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\Omega \times Y} u(x,y)\varphi(x,y)\, \mathrm{d}x \ \mathrm{d}y.$$

This yields a separation between the macroscopic and microscopic scales, the oscillations of u^ε being encoded in the variable y of u.

Theorem

Every bounded sequence in $L^2(\Omega)$ admits a two-scale converging subsequence.

Theorem

There exists $u \in H_0^1(\Omega)$ *and* $\widehat{u} = \widehat{u}(x,y) \in L^2(\Omega; V_\#)$ *such that:*

- u^ε *converges to* u *weakly in* $H_0^1(\Omega)$ *and strongly in* $L^2(\Omega)$.
- ∇u^ε *two-scale converges to* $\nabla u + \nabla_y \widehat{u}$ *in* $\left[L^2(\Omega \times Y)\right]^2$.
- *The pair* $(u, \widehat{u})$ *solves the two-scale limit problem:*

$$\int_{\Omega\times Y} a(y)\Big[\nabla u(x)+\nabla_y \widehat{u}(x,y)\Big]\cdot\Big[\nabla v+\nabla_y \widehat{v}(x,y)\Big]\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f v,$$

for all $v \in H_0^1(\Omega)$ *and* $\widehat{v} \in L^2(\Omega; V_\#)$.

Outline

Theorem

Assume that

$$\kappa > \kappa_Y \qquad \text{or} \qquad \kappa < 1/\kappa'_Y.$$

Then, the homogenized problem

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -\mathrm{div}\left(a^H \nabla u\right) & = & f, & \text{in } \Omega \\ u & = & 0, & \text{on } \Gamma, \end{array} \right.$$

is well-posed in $H_0^1(\Omega)$.

PROOF (Outline)

- We first prove that the two-scale limit problem is well-posed.

- We (classically) show that the two-scale limit problem admits an equivalent uncoupled formulation

  - the homogenized problem $(\mathcal{P})$ defined in Ω for u

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -\mathrm{div}\left(a^H \nabla u\right) & = & f, & \text{in } \Omega \\ u & = & 0, & \text{on } \Gamma, \end{array} \right.$$

  - an explicit expression for $\widehat{u}$ involving the cell problems in Y:

$$\widehat{u}(x,y) = \chi_1(y)\frac{\partial u}{\partial x_1}(x) + \chi_2(y)\frac{\partial u}{\partial x_2}(x).$$

- We conclude to the well-posedness of $(\mathcal{P})$.

PROOF (Well-posedness of the 2-scale limit problem)

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega; V_\#)$$

$$\begin{aligned}\|\mathcal{U}\|_{\mathcal{H}} : &= \left\{\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla_y \widehat{u}\|^2_{L^2(\Omega\times Y)}\right\}^{\frac{1}{2}}, \quad \forall \mathcal{U} = (u, \widehat{u}) \in \mathcal{H} \\ &= \|\nabla u + \nabla_y \widehat{u}\|_{L^2(\Omega\times Y)}.\end{aligned}$$

The **two-scale limit problem** reads:

Find $\mathcal{U} = (u, \widehat{u}) \in \mathcal{H}$ such that

$$\mathcal{B}(\mathcal{U}, \mathcal{V}) = \int_\Omega f v, \qquad \forall\, \mathcal{V} = (v, \widehat{v}) \in \mathcal{H},$$

where

$$\mathcal{B}(\mathcal{U}, \mathcal{V}) := \int_\Omega \int_Y a(y) \Big[\nabla u(x) + \nabla_y \widehat{u}(x,y)\Big] \cdot \Big[\nabla v + \nabla_y \widehat{v}(x,y)\Big]\, \mathrm{d}x\, \mathrm{d}y,$$

Proof (Large Contrasts)

$$\mathcal{T}\mathcal{U} = (u, \mathbf{T}\widehat{u}), \quad \forall \mathcal{U} = (u, \widehat{u}) \in \mathcal{H},$$

with: $\mathbf{T}u := \widetilde{\mathbf{T}}u - \left(\int_Y \widetilde{\mathbf{T}}u\right), \quad \widetilde{\mathbf{T}}u = \begin{cases} u_1 & \text{in } Y_1 \\ -u_2 + 2\mathbf{P}(u_1) & \text{in } Y_2. \end{cases}$

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$$\mathcal{B}(\mathcal{U}, \mathcal{T}\mathcal{U}) = \int_{\Omega\times Y} a(y)\Big[\nabla u(x) + \nabla_y \widehat{u}(x,y)\Big] \cdot \Big[\nabla u(x) + \nabla_y \mathbf{T}\widehat{u}(x,y)\Big]\, \mathrm{d}x\, \mathrm{d}y,$$

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Setting $\overline{y} = y - \mathcal{M}_Y(y)$, we have

$$\nabla u(x) + \nabla_y \widehat{u}(x,y) = \nabla_y \Big[\nabla u(x) \cdot \overline{y} + \widehat{u}(x,y)\Big] = \nabla_y U_x(y)$$

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$$\begin{aligned} \nabla u(x) + \nabla_y \mathbf{T}\widehat{u}(x,y) &= \nabla_y \Big[\nabla u(x) \cdot \overline{y} + \mathbf{T}\widehat{u}(x,y)\Big] \\ &= \nabla_y \Big[\nabla u(x) \cdot \mathbf{T}\overline{y} + \mathbf{T}\widehat{u}(x,y)\Big] \\ &= \nabla_y \left(\mathbf{T}U_x(y)\right) \end{aligned}$$

Proof (Large Contrasts)

$$\begin{aligned}
\mathcal{B}(\mathcal{U}, \mathcal{T}\mathcal{U}) &= \int_{\Omega\times Y} a(y)\Big[\nabla u(x) + \nabla_y \widehat{u}(x,y)\Big] \cdot \Big[\nabla u(x) + \nabla_y \mathbf{T}\widehat{u}(x,y)\Big]\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_\Omega \left(\int_Y a(y) \nabla_y U_x(y) \cdot \nabla_y \left(\mathbf{T} U_x(y)\right)\, \mathrm{d}y \right) \mathrm{d}x, \\
&\geqslant \gamma \int_\Omega \|\nabla_y U_x\|^2_{L^2(Y)}\, \mathrm{d}x \\
&= \gamma \|\mathcal{U}\|^2_{\mathcal{H}}, \qquad \forall \kappa > \kappa'_Y.
\end{aligned}$$

Proof (Small Contrasts)

$$\mathbf{T}u := \widetilde{\mathbf{T}}u - \left(\int_Y \widetilde{\mathbf{T}}u\right),$$

with

$$\widetilde{\mathbf{T}}u = \begin{cases} u_1 - 2\mathbf{Q}(u_2 - \mathcal{M}_2(u_2)) & \text{in } Y_1 \\ -u_2 + 2\mathcal{M}_2(u_2) & \text{in } Y_2, \end{cases}$$

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⚠

Affine functions are not invariant by the harmonic extension $\mathbf{Q}$ from $H^1(Y_2)$ to $H^1_0(Y)$, and hence by $\mathbf{T}$.

Proof (Small Contrasts)

For all $\mathcal{U} = (u, \widehat{u}) \in \mathcal{H}$, set:

$$\mathcal{T}\mathcal{U} = (u, \widehat{v}), \qquad \widehat{v} := \mathbf{T}\widehat{u} + \nabla u(x) \cdot [\mathbf{T}\overline{y} - \overline{y}]$$

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$$\mathcal{B}(\mathcal{U}, \mathcal{T}\mathcal{U}) = \int_{\Omega\times Y} a(y)\Big[\underbrace{\nabla u(x) \cdot \overline{y} + \widehat{u}(x,y)}_{U_x(y)}\Big] \cdot \Big[\nabla u(x) + \nabla_y \widehat{v}(x,y)\Big]\, \mathrm{d}x\, \mathrm{d}y,$$

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$$\begin{aligned}
\nabla_y \widehat{v}(x,y) &= \nabla_y \left(\mathbf{T}\widehat{u} + \nabla u(x) \cdot \mathbf{T}\overline{y} \right) - \nabla_y (\nabla u(x) \cdot \overline{y}) \\
&= \nabla_y \left(\mathbf{T}\widehat{u} + \nabla u(x) \cdot \mathbf{T}\overline{y} \right) - \nabla u(x) \\
&= \nabla_y \left(\mathbf{T} U_x \right) - \nabla u(x).
\end{aligned}$$

PROOF (Small Contrasts)

For all $\mathcal{U} = (u, \widehat{u}) \in \mathcal{H}$, set:

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$$\begin{aligned} \nabla_y \widehat{v}(x,y) &= \nabla_y \left(\mathbf{T}\widehat{u} + \nabla u(x) \cdot \mathbf{T}\overline{y} \right) - \nabla_y (\nabla u(x) \cdot \overline{y}) \\ &= \nabla_y \left(\mathbf{T}\widehat{u} + \nabla u(x) \cdot \mathbf{T}\overline{y} \right) - \nabla u(x) \\ &= \nabla_y \left(\mathbf{T} U_x \right) - \nabla u(x). \end{aligned}$$

$$\begin{aligned} \mathcal{B}(\mathcal{U}, \mathcal{T}\mathcal{U}) &= \int_{\Omega \times Y} a(y) \nabla_y U_x(y) \cdot \nabla_y (\mathbf{T} U_x)(y) \, \mathrm{d}x \, \mathrm{d}y, \\ &\geqslant \gamma' \int_\Omega \| \nabla_y U_x \|^2_{L^2(Y)} \, \mathrm{d}x, \\ &= \gamma' \| \mathcal{U} \|^2_{\mathcal{H}}. \end{aligned}$$

■

Concluding comments

1. The matrix a^H is positive definite for $\kappa > \kappa_Y$ and $\kappa < 1/\kappa'_Y$.

2. For $1/\kappa'_Y < \kappa < \kappa_Y$ and smooth interface ∂Y_2, $(\mathcal{P}^\varepsilon)$ is well-posed if and only if

$$\kappa \notin \{\kappa_n^\varepsilon, n \geqslant 1\},$$

   where (κ_n^ε) is a sequence tending to 1. It is not clear whether there exists or not (and under what conditions on the geometry) a non empty subset of the critical interval which is uniformly free of the values κ_n^ε as $\varepsilon \to 0$.

3. We have extended these results to other operators (scalar systems with extreme contrasts, Maxwell's equations) and other geometries (thin periodic domains).

Happy Birthday

Marius !

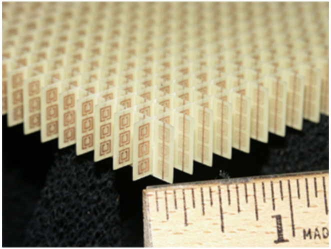

Figure: A split-ring resonator array arranged to produce a negative index of refraction (Wikipedia).

Metamaterials

Metamaterials (also called negative or left-handed materials) are artificial composite materials exhibiting **negative dielectric permittivity and magnetic permeability** over some range of frequencies, and hence behaving as **negative refractive index materials** (Victor Veselago in 1967, John Pendry in the late 90's).
Optical Applications : superlens, cloaking, biomedical imaging...

BACK