Boundary Controllability as Limit of Distributed Controllability

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Marius 60
Bordeaux 2021 .

in honor of Marius

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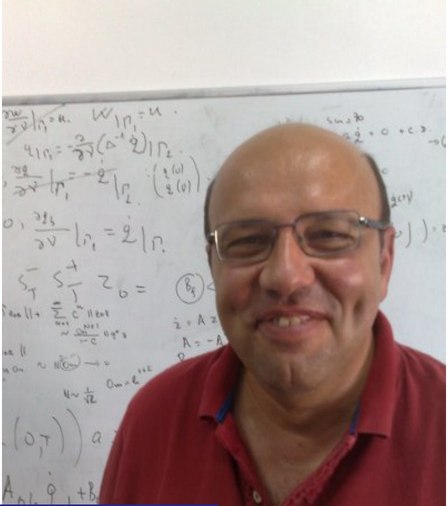


Marius ... a little younger...

The subject of this presentation is a good opportunity to dedicate also this work to the memory of

Caroline Fabre.

who passed away in 2020 ...

Outline

Distributed controllability problem with control on an ϵ-neighborhood of a part of the boundary.
What happens when $\epsilon \to 0$? Do we have "convergence" to a boundary controllability problem ?
Notations :
Ω is a bounded regular open set of R^N with boundary Γ. $Q = \Omega \times (0, T)$.
Γ_0 is a non empty open subset of Γ and

$$\omega_\epsilon = \bigcup_{x \in \Gamma_0} (B(x.\epsilon) \cap \Omega).$$

For the wave equation we worked on this question in the early 90's together with Caroline Fabre.

• C. Fabre, J.-P. Puel, Behavior near the boundary for solutions of the wave equation. J. Differential Equations, 106 (1993), no. 1, 186–213.

• C. Fabre, Exact boundary controllability of the wave equation as the limit of internal controllability. SIAM J. Control Optim., 30 (1992), no. 5, 1066–1086.

Recently
Collaboration with Felipe Chaves-Silva and Mauricio Cardoso-Santos
we worked on the case of the heat equation which had not been considered before.

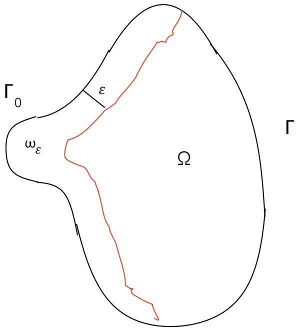
• F.W. Chaves-Silva, J.-P. Puel and M.C. Santos, Boundary null controllability as the limit of internal controllability: The heat case. ESAIM: COCV 26 (2020) 91.

Problem for the wave equation

$$
\begin{aligned}
&\partial_{tt} y - \Delta y = v.\mathbf{1}_{\omega_\epsilon} \text{ in } \Omega \times (0, T),\\
&y = 0 \text{ on } \Gamma \times (0, T),\\
&y(0, x) = y_0(x),\ \partial_t y(0, x) = y_1(x) \text{ in } \Omega,\\
&y(T, x) = 0,\ \partial_t y(T, x) = 0 \text{ in } \Omega.
\end{aligned}
$$

In order to solve this problem we need a geometric condition (cf. Bardos-Lebeau-Rauch) saying essentially that every ray of the geometric optics, propagating at speed 1 meets ω_ϵ before time T.

Case of the wave equation



Case of the wave equation

Using Lions' H.U.M. we find a solution for $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$ by considering the adjoint equation

$$\begin{aligned}
&\partial_{tt}\varphi_\epsilon - \Delta y \varphi_\epsilon = 0 \text{ in } \Omega \times (0,T),\\
&\varphi_\epsilon = 0 \text{ on } \Gamma \times (0,T),\\
&\varphi_\epsilon(0,x) = \varphi_{\epsilon 0}(x),\ \partial_t\varphi_\epsilon(0,x) = \varphi_{\epsilon 1}(x) \text{ in } \Omega,
\end{aligned}$$

for $\varphi_{\epsilon 0} \in L^2(\Omega)$ and $\varphi_{\epsilon 1} \in H^{-1}(\Omega)$ suitably chosen thanks to the observability inequality

$$|\varphi_{\epsilon 0}|^2_{L^2(\Omega)} + ||\varphi_{\epsilon 1}||^2_{H^{-1}(\Omega)} \leq C(\epsilon) \int_{\omega_\epsilon \times (0,T)} |\varphi_\epsilon(x,t)|^2 dxdt.$$

Case of the wave equation. Passage to the limit

No hope to obtain classical estimates as the boundary condition cannot be preserved.

Three main difficulties.

- We can show that when we have

$$\frac{1}{\epsilon^3}\int_{\omega_\epsilon\times(0,T)}|\varphi_\epsilon(x,t)|^2 dxdt \leq C$$

where C is independent of ϵ, and when φ_ϵ coverges weakly to φ (in the natural spaces), then at the limit

$$\frac{\partial\varphi}{\partial\nu} \in L^2(\Gamma_0 \times (0,T),)$$

and if (Γ_0, T) satisfies the geometric condition then φ is a solution of finite energy and the initial conditions at the limit verify $(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Case of the wave equation. Passage to the limit

• Then we show that the constant $C(\epsilon)$ satisfies

$$C(\epsilon) = O(\epsilon^{-3}).$$

This is a key point to obtain an estimate for the control.
• We then have to pass to the limit in a weak formulation of the wave equation to obtain at the limit the following boundary controllability system.

$$\begin{aligned}
&\partial_{tt} y - \Delta y = 0 \text{ in } \Omega \times (0, T),\\
&y = -\frac{1}{3}\frac{\partial \varphi}{\partial \nu} \text{ on } \Gamma_0 \times (0, T),\\
&y = 0 \text{ on } (\Gamma \setminus \Gamma_0) \times (0, T),\\
&y(0, x) = y_0(x),\ \partial_t y(0, x) = y_1(x) \text{ in } \Omega,\\
&y(T, x) = 0,\ \partial_t y(T, x) = 0 \text{ in } \Omega,
\end{aligned}$$

where φ is solution of the adjoint problem at the limit.

Case of the heat equation

$$\begin{aligned}
&\partial_t y - \Delta y = v_\epsilon . \mathbb{1}_{\omega_\epsilon} \text{ in } \Omega \times (0, T),\\
&y = 0 \text{ on } \Gamma \times (0, T),\\
&y(0, x) = y_0(x), \ \text{in}\, \Omega,\\
&y(T, x) = 0, \ \text{in}\, \Omega.
\end{aligned}$$

Here we do not need any geometric condition and it is by now classical that this problem has a solution for any $T > 0$.

We solve this problem thanks to the following observability inequality

$$|\varphi(0)|^2_{L^2(\Omega)} \leq C(\epsilon) \int_{\omega_\epsilon \times (0,T)} |\varphi|^2 dxdt,$$

where φ is solution of the adjoint problem

$$\begin{aligned}
&-\partial_t \varphi - \Delta \varphi = 0 \text{ in } \Omega \times (0, T),\\
&\varphi = 0 \text{ on } \Gamma \times (0, T),\\
&\varphi(T, x) = \varphi_T, \ \text{in}\, \Omega.
\end{aligned}$$

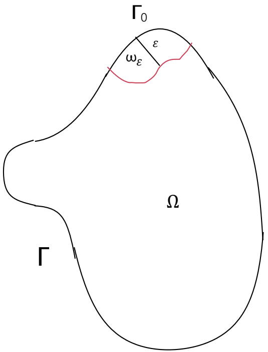
Case of the heat equation

Difficulties :

• The usual way to obtain this inequality is to use a global Carleman estimate with a principal positive regular weight having zero boundary condition and a non vanishing gradient in $\Omega \setminus \omega_\epsilon$, therefore a very steep gradient in ω_ϵ.

• There is no hope to obtain estimates of $C(\epsilon)$ directly with this method. We will show that $C(\epsilon) = O(\epsilon^{-3})$ and this is sharp and is the major difficulty of this work.

• Passage to the limit which is treated in an analogous way as for the case of the wave equation.


Γ_0
ε
ω_ε
Ω
Γ

Sharp Carleman estimate-Weights

We start with a basic weight $\psi \in C^2(\bar{\Omega})$ such that

$$|\nabla\psi(x)| > 0\,\forall x \in \bar{\Omega}, \text{ and } \frac{\partial\psi}{\partial\nu}(x) \leq 0\,\forall x \in \Gamma \setminus \Gamma_0.$$

Then for $\lambda > 0$ we set

$$\phi(x,t) = \frac{e^{\lambda(\psi(x)+m_1)}}{t(T-t)},\ \alpha(x,t) = \frac{e^{\lambda(\psi(x)+m_1)} - e^{\lambda(||\psi||_\infty + m_2)}}{t(T-t)},$$

where for example $m_1 = ||\psi||_\infty + 2$ and $m_2 = ||\psi||_\infty + 3$. Then α is negative and for $s > 0$ terms like $e^{2s\alpha}\phi^k$ are uniformly bounded.

Sharp Carleman estimate

We introduce the notation

$$I(s,\lambda,\varphi) = s^3\lambda^4 \int_Q e^{2s\alpha}\phi^3|\varphi|^2 dxdt + s\lambda^2 \int_Q e^{2s\alpha}\phi|\nabla\varphi|^2 dxdt$$
$$+s^{-1}\int_Q e^{2s\alpha}\phi^{-1}(|\varphi_t|^2 + \sum_{i,j=1}^N |\frac{\partial^2\varphi}{\partial x_i \partial x_j}|^2)dxdt$$

• We begin with a boundary Carleman estimate
There exists C such that for s and λ large enough

$$I(s,\lambda,\varphi) \leq Cs\lambda \int_{\Gamma_0\times(0,T)} e^{2s\alpha}\phi|\frac{\partial\varphi}{\partial\nu}|^2 d\gamma dt.$$

This is relatively classical.

Sharp Carleman estimate

• Then we prove that for any $\delta > 0$ we can fin $C(\delta)$ such that

$$s\lambda \int_{\Gamma_0\times(0,T)} e^{2s\alpha}\phi|\frac{\partial\varphi}{\partial\nu}|^2 d\gamma dt \leq C(\delta)\epsilon^{-3}s^7\lambda^4 \int_{\omega_\epsilon\times(0,T)} e^{2s\alpha}\phi^7|\varphi|^2 dxdt + \delta I(s,\lambda,\varphi).$$

This part is long and very technical. It requires a covering of the boundary, transformation to local coordinates and very sharp estimates.

• Putting this together we obtain the following Carleman estimate

$$I(s,\lambda,\varphi) \leq C\epsilon^{-3}s^7\lambda^4 \int_{\omega_\epsilon\times(0,T)} e^{2s\alpha}\phi^7|\varphi|^2 dxdt.$$

Observability inequality

Let us define

$$I(t) = \frac{T^2}{4} \text{ if } 0 \leq t \leq \frac{T}{2}, \; I(t) = t(T-t) \text{ if } \frac{T}{2} \leq t \leq T.$$

And then

$$\gamma(x,t) = \frac{e^{\lambda(\psi(x)+m_1)}}{I(t)}, \; \beta(x,t) = \frac{e^{\lambda(\psi(x)+m_1)} - e^{\lambda(||\psi||_\infty + m_2)}}{I(t)},$$

We now use the notation

$$J(s,\lambda,\varphi) = s^3\lambda^4 \int_Q e^{2s\beta}\gamma^3|\varphi|^2 dxdt + s\lambda^2 \int_Q e^{2s\beta}\gamma|\nabla\varphi|^2 dxdt$$
$$+ s^{-1}\int_Q e^{2s\beta}\gamma^{-1}(|\varphi_t|^2 + \sum_{i,j=1}^{N} |\frac{\partial^2\varphi}{\partial x_i \partial x_j}|)dxdt$$

Using classical energy estimates with a cut-off function we deduce the observability inequality (again for s and λ large enough)

$$|\varphi(0)|^2_{L^2(\Omega)} + J(s,\lambda,\varphi) \leq C\epsilon^{-3}s^7 \int_{\omega_\epsilon \times (0,T)} e^{2s\beta}\gamma^7|\varphi|^2 dxdt.$$

Construction of a sequence of controls

In order to simplify take

$$\gamma^*(t) = \max_{x\in\Omega} \gamma(x,t), \text{ and } \beta^*(t) = \max_{x\in\Omega} \beta(x,t).$$

This enables us to find, for the distributed controllability problem, a control v_ϵ of the form

$$v_\epsilon = \frac{1}{\epsilon^3} e^{2s\beta^*} {\gamma^*}^7 \varphi_\epsilon,$$

where φ_ϵ is a solution of the adjoint problem suitably chosen, which converges weakly to φ also solution of the adjoint problem.

Passage to the limit

We then have

$$\partial_t y_\epsilon - \Delta y_\epsilon = \frac{1}{\epsilon^3} e^{2s\beta^*} \gamma^{*7} \varphi_\epsilon . \mathbb{1}_{\omega_\epsilon} \text{ in } \Omega \times (0,T),$$
$$y_\epsilon = 0 \text{ on } \Gamma \times (0,T),$$
$$y_\epsilon(0,x) = y_0(x), \ \text{ in } \Omega,$$
$$y_\epsilon(T,x) = 0, \ \text{ in } \Omega.$$

The construction of φ_ϵ implies

$$\frac{1}{\epsilon^3} \int_{\omega_\epsilon \times (0,T)} e^{2s\beta^*} \gamma^{*7} |\varphi_\epsilon|^2 dxdt \leq C |y_0|^2_{L^2(\Omega)},$$

and the convergence

$$e^{s\beta^*} \gamma^{*\frac{7}{2}} \varphi_\epsilon \to e^{s\beta^*} \gamma^{*\frac{7}{2}} \varphi.$$

We have a heat equation with a right hand side concentrated and becoming singular.
We then proceed as for the case of the wave equation in a quite analogous way by passing to the limit in the weak formulation of the controlled equation and we obtain at the limit the boundary control problem

$$
\begin{aligned}
&\partial_t y - \Delta y = 0 \text{ in } \Omega \times (0, T),\\
&y = -\frac{1}{3} e^{2s\beta^*} \gamma^{*7} \frac{\partial \varphi}{\partial \nu} \text{ on } \Gamma_0 \times (0, T),\\
&y = 0 \text{ on } (\Gamma \setminus \Gamma_0) \times (0, T),\\
&y(0, x) = y_0(x), \ \text{ in } \Omega,\\
&y(T, x) = 0, \ \text{ in } \Omega.
\end{aligned}
$$

Viata lunga Marius

Thank you for your attention