Large time behaviour of a fluid-structure interaction model

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Control and Analysis of PDE systems, in honour of Marius Tucsnak for his 60th birthday, Bordeaux

Joint works with Sylvain Ervedoza and Marius Tucsnak.
1 1D FSI models.

2 3D incompressible NS + Rigid body
1 1D FSI models.

2 3D incompressible NS + Rigid body
1D simplified model

- We consider a one dimensional model for the motion of a particle (piston) in a cylinder filled with a viscous fluid.

- The extremities of the cylinder are fixed in the case of BVP.

- The fluid is modelled by the 1D viscous Burgers equation, whereas the piston obeys Newton’s second law.

- fluid-piston system evolves in the interval $(-1, 1)$ or $\mathbb{R}$ and $h : [0, \infty) \mapsto (-1, 1)$ (or $\mathbb{R}$) denotes the position of the particle.
Governing equations

**Motion of the gas:**
Described in the Eulerian coordinate system by its velocity \( u = u(t, x) \), which satisfy the one dimensional Burgers equation

\[
\partial_t u + u \partial_x u - \partial_{xx} u = 0, \quad t \geq 0, x \neq h(t)
\]

\((BVP) \sim u(-1, t) = 0 = u(1, t)\)

**Motion of the Piston:** Described by Newton's law:

\[
m \ddot{h}(t) = [\partial_x u](t, h(t)) \quad (t \geq 0),
\]

where \( m \) is the mass of the piston and the symbol \([f](t, x)\) stands for the jump at instant \( t \) of \( f \) at \( x \), i.e.,

\[
[f](t, x) = f(t, x^+) - f(t, x^-).
\]

**Equality of the velocity:**

\[
u(t, h(t)) = \dot{h}(t) \quad (t \geq 0),
\]

The position of the piston (and, consequently, the domain occupied by the gas) is one of the unknowns of the problem, we have a free boundary value problem.
Goal

• Existence and uniqueness of solutions.
• Large time behaviour of the point particle:
  • $h(t)$ remains bounded for all time.
  • Particle escapes to spatial infinity.
  • BVP $\sim$ (No) contact with the fluid boundary.
• Control and Stabilization problem: move the point particle from one point to another.

Well-posedness: Vázquez and Zuazua (03, 05),
Control and Stabilization: Doubova and Fernández-Cara (05), Liu, Takahashi and Tucsnak (13), Cîndea et. al. (15), Ramaswamy, Roy and Takahashi (20), Badra and Takahashi (14)
Cauchy problem

• For any \((u_0, \ell_0) \in L^2(\mathbb{R}) \times \mathbb{R}\), the system admits a unique global solution \(u \in C([0, \infty); L^2(\mathbb{R}))\), \(h \in C^1([0, \infty))\).

• \(u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})\) and \(M = \int_{\mathbb{R}\setminus\{0\}} u_0 \, dx + m\ell_0 \neq 0\), then

\[
\|u(t)\|_{L^r(\mathbb{R})} \lesssim t^{-1/2(1-1/r)}, \quad t > 0, \ r \geq 1.
\]

\[
\frac{1}{c}(1 + t)^{-1/2} < h'(t) < c(1 + t)^{-1/2},
\]

for some \(c \geq 1\), and \(c\) is explicit in terms of \(M\).

• Thus the particle escapes to spatial infinity: \(\int_0^\infty (1 + t)^{-1/2} = \infty\).
• For any $h_0 \in (-1, 1)$, $(u_0, \ell_0) \in H^1_0(-1, 1)$ with $u_0(h_0) = \ell_0$ the system admits a unique global strong solution

$$u \in L^2(0, T; H^2) \cap H^1(0, T; L^2), \quad h \in H^2(0, T),$$

(No contact) $h(t) \in (-1, 1)$ for all $t \geq 0$.

• We also have

$$\|u(t)\|_{L^2} + |h'(t)| \lesssim e^{-\gamma t} \text{ and } \lim_{t \to \infty} h(t) = h_* \in (-1, 1).$$

• What is $h_*$?
1D piston problem

- Fluid: Compressible Navier-Stokes.
- \( \rho \): density of the fluid, \( u \): velocity of the fluid.

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 & t > 0, \ x \in \mathbb{R} \setminus \{h(t)\}, \\
\rho (\partial_t u + u \partial_x u) - \partial_{xx} u + \partial_x (\rho \gamma) &= 0 & t > 0, \ x \in \mathbb{R} \setminus \{h(t)\}, \\
u(t, h(t) \pm 0) &= h'(t) & t > 0, \\
mh''(t) &= [\partial_x u - \rho \gamma](t, h(t)) & t > 0,
\end{align*}
\]

+ initial data

- Bounded domain - fluid domain: \((-1, 1) \setminus \{h(t)\}\) and \(u(t, \pm 1) = 0\).
1D piston problem

• **Unbounded domain**: Koike (2020)

\[ |h'(t)| \lesssim (1 + t)^{-3/2} \implies |h(t) - h_0| \leq C. \]

• **Bounded domain**: Shelukhin (1977, 1982)

\[ \|u(t)\|_2 \to 0, \quad \|\rho(t) - \bar{\rho}\|_2 \to 0, \quad h'(t) \to 0 \]

\[
\lim_{t \to \infty} h(t) = \frac{M_L - M_R}{M_L + M_R}, \quad M_L = \int_{-1}^{h_0} \rho_0, \quad M_M = \int_{h_0}^{1} \rho_0,
\]

• **Adiabatic piston in bounded domain**: Feireisl et al. (18)

\[
\sim \lim_{t \to \infty} h(t) = h_*, \text{ and } h_* \text{ is unknown.}
\]
Let us consider fluid rigid body problem in whole $\mathbb{R}^3$, where fluid is incompressible Navier-Stokes and the rigid body is a ball.

**Theorem**

If the initial datum is small enough in suitable norm then the position of the center of the rigid ball converges to some $h_* \in \mathbb{R}^3$ as time goes to infinity.

\cite{Ervedoza2023}
Outline

1. 1D FSI models.
2. 3D incompressible NS + Rigid body
3D problem

- Fluid + rigid body = $\Omega \subset \mathbb{R}^3$ or Fluid + rigid body = $\mathbb{R}^3$.
- $\Omega_s(t)$ ~ the domain occupied by the solid at time $t \geq 0$.
- $\text{dist} (\Omega_s(0), \partial \Omega) \geq \nu > 0$.
- $h(t) \in \mathbb{R}^3$ ~ the position of the center of mass,
  $Q(t) \in SO_3(\mathbb{R})$ ~ the orthogonal matrix giving the orientation of the solid,
  $\omega(t) \in \mathbb{R}^3$ ~ the angular velocity of the rigid body.
- $\Omega_s(t) = h(t) + Q(t)y$, $y \in \Omega_s(0)$,
- Fluid domain: $\Omega_f(t) = \Omega \setminus \Omega_s(t)$ or $\Omega_f(t) = \mathbb{R}^3 \setminus \Omega_s(t)$
Governed equations:

- **Fluid equation:** Incompressible Navier-Stokes-Fourier equations:

  For \((t, x) \in (0, \infty) \times \Omega_f(t),\)

  \[
  \rho_f (\partial_t u + u \cdot \nabla u) - \text{div} \sigma(u, \pi) = 0, \quad \text{div} \ u = 0, \\
  \sigma(u, \pi) = \nu(\nabla u + \nabla u^\top) - \pi I_3, \\
  u(t, x) = h'(t) + \omega(t) \times (x - h(t)) \quad x \in \partial\Omega_s(t), \\
  mh''(t) = -\int_{\partial\Omega_s(t)} \sigma(u, p)n \, d\gamma, \\
  J\omega'(t) = (J\omega) \times \omega - \int_{\partial\Omega_s(t)} (x - h(t)) \times \sigma(u, p)n \, d\gamma, \\
  + \text{ initial data}
  \]

In bounded domain \(u = 0\) on \(\partial\Omega\).
Existence and uniqueness

∼ Serre(1987), Takahashi (02), Feireisl (03), Takahashi and Tucsnak(04), Cumsille and Takahashi (08), Geissert et. al (13) and many more..

• Global existence of weak solutions.
• Global existence of strong solution in 2D.
• Local in time or global in time for small data in the 3D case.
• Similar results for bounded domain.
Large time behaviour : bounded domain

\sim \text{Takahashi (04), Maity and Tucsnak (18)}

- For initial data sufficiently small
  \[ \| u(t) \|_{L^q} + |h'(t)| + |\omega(t)| \lesssim e^{-\eta t}, \]
  for some \( \eta > 0 \).
- \( q \in (4/3, \infty) \) in 2D and \( q \in (5/3, \infty) \) in 3D.
- \( h(t) \) remains away from the boundary, \( \lim_{t \to \infty} h(t) = h_* \).
- We don’t know what is \( h_* \).
Large time behaviour: 2D unbounded domain

\sim \text{Ervedoza, Hillairet and Lacave (2014)}

- Rigid body is a ball.
- For initial data sufficiently small

\[ \|u(t)\|_{L^2} \to 0, \quad |h'(t)| \lesssim \frac{1}{t}. \]

- Possible unbounded trajectory for the rigid ball. In fact, for the linearized problem, both bounded and unbounded trajectories are possible for the rigid ball, depending on the “mass:” \((m - \pi)\ell_0\).
Our result: 3D unbounded domain and rigid body is a ball

Theorem

If the initial datum is small enough in suitable norm then the position of the center of the rigid ball converges to some \( h_* \in \mathbb{R}^3 \) as time goes to infinity.

\[ \sim \] Dimension plays a role. Heuristically, \( L^\infty \) norm of velocity, for initial data in \( L^q \), decays like \( \sim t^{-n/2q} \).
Why we need rigid body to be a ball

- Reformulate the problem in a fixed domain, decay estimates for the linearized problem and a suitable fixed point.
- Use the change of variable \( x \rightarrow Q(t)x + h(t) \). More precisely,

\[
\begin{align*}
v(t) &= Q(t)^{\top} u(t, Q(t)x + h(t)), \\
\ell(t) &= Q(t)^{\top} h(t), \\
\omega(t) &= Q(t)^{\top} \omega(t)
\end{align*}
\]
Transformed system: in \((0, \infty) \times \Omega_f(0), \Omega_f(0) = \mathbb{R}^3 \setminus \Omega_s(0)\):

\[
\partial_t \nu - \text{div} \sigma(\nu, \pi) = -(\nu - \ell) \cdot \nabla \nu + (\omega \times y) \cdot \nabla \nu, \quad \text{div} \nu = 0,
\]

\[
\nu(t, y) = \ell(t) + \omega(t) \times y, \quad y \in \partial \Omega_s(0),
\]

\[
m \ell'(t) = -\int_{\partial \Omega_s(0)} \sigma(\nu, \pi)n \, d\gamma,
\]

\[
J \omega'(t) = -\int_{\partial \Omega_s(0)} y \times \sigma(\nu, \pi)n \, d\gamma,
\]

+ initial data

\sim \text{if the rigid body is not a ball, simple change of variable gives the spatial unbounded term}: (\omega \times y) \cdot \nabla u

\sim \text{We need more complicated change of variable, which induces a lot of nonlinear term.}
Mild solutions

We set

\[ V = \begin{cases} \nu & \text{in } \mathbb{R}^3 \setminus B(0,1) \\ \ell + \omega \times y & \text{in } B(0,1). \end{cases} \]

and, for \( 1 < q < \infty \),

\[ X^q = \{ \Phi \in L^q(\mathbb{R}^3)^3 \mid \text{div} \Phi = 0 \text{ in } \mathbb{R}^3, D(\Phi) = 0 \text{ in } B(0,1) \} \]

Then we can rewrite the system

\[ V'(t) = A_q V(t) + F(V), \quad V(0) = V_0, \]

where \( A_q : D(A_q) \to X^q \) is the linear fluid-structure operator and \( F \) nonlinear terms.

\[ V(t) = T_t V_0 + \int_0^t T_{t-s} F(V(s)) \]
Our result

- Existence and uniqueness: For $v_0 \in L^3$, $\text{div } v_0 = 0$ satisfying compatibility conditions and

$$\|v_0\|_{L^3} + |\ell_0| + |\omega_0| \ll 1,$$

the system admits a unique solution in $C([0, \infty); L^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$.

- Decay estimates: Let $q \in (1, 3/2)$, $v_0 \in L^q \cap L^3$ with compatibility conditions and same smallness assumption. Then

$$\|u(t)\|_{L^\infty} + |h'(t)| + |\omega(t)| \lesssim t^{-3/2q}$$

$\implies h(t)$ converges to $h_*$. We don’t know what is $h_*$. 
Linear FSI system with arbitrary rigid body

\[ E - \text{Fluid domain, } O - \text{rigid body.} \]

\[
\begin{align*}
\partial_t u - \mu \Delta u + \nabla \pi &= 0, \quad \text{div} u = 0 \quad (t \geq 0, x \in E), \\
u &= \ell + \omega \times x \quad (t \geq 0, x \in \partial O), \\
m \dot{\ell} + \int_{\partial O} \sigma(u, \pi) \nu \, ds &= 0 \quad (t \geq 0), \\
J \dot{\omega} + \int_{\partial O} x \times \sigma(u, \pi) \nu \, ds &= 0, \quad (t \geq 0), \\
u(0) &= u_0 \quad (x \in E), \\
\ell(0) &= \ell_0, \quad \omega(0) = \omega_0.
\end{align*}
\]

We rewrite the system as

\[
\frac{d}{dt} U = A_q U, \quad U(0) = U_0.
\]
Decay estimates for FSI semigroup

Theorem (SE, DM, MT)

- **1 < q ≤ r < ∞ or 1 < q < ∞, r = ∞. Then**

\[
\| \mathbb{T}_t U \|_r \leq C(q, r) t^{-3/2(1/q - 1/r)} \| U \|_q \quad t > 0, U \in \mathbb{X}^q.
\]

- **1 < q ≤ r ≤ 3. Then**

\[
\| \nabla \mathbb{T}_t U \|_{r, E} \leq C(q, r) t^{-3/2(1/q - 1/r) - 1/2} \| U \|_q \quad t > 0, U \in \mathbb{X}^q.
\]

~ The decay estimates are same as Stokes system in the exterior domain (Iwashita, 1989).

~ Tools: Two types of resolvent estimates and some cut-off arguments.
Theorem

Let $1 < q < \infty$. Then

$$\| \lambda (\lambda I - A_q)^{-1} \| \leq C, \text{ for } \lambda \in \Sigma_\theta, \theta > \pi/2.$$

- $A_2$ is self adjoint $\implies$ $A_2$ is sectorial. (Takahashi and Tucsnak, 2004).
- $A$ is sectorial on $X^2 \cap X^q$, $q \geq 6$. (Wang and Xin, 2011).
- in bounded domain, $A_q$ is sectorial for any $1 < q < \infty$. (Maity and Tucsnak, 18).
- Idea : Resolvent of FSI = Resolvent of Stokes + “other terms”
Small time decay estimates

• Consequence of the resolvent estimate:
  \[ \| A_q^m T_t \| \lesssim t^{-m} \text{ for any } m \in \mathbb{N}. \]

• A priori estimate:
  \[ \| u \|_{2m,q} + |\ell| + |\omega| \lesssim \| A_q^m (u, \ell, \omega) \|_{X_q} + \| (u, \ell, \omega) \|_{X_q} \]

• \( N = 3[1/q - 1/r] \). Assume \( N \) is even. Then
  \[
  \| u(t) \|_{N,q} + |\ell(t)| + |\omega(t)| \lesssim \| A_q^{N/2} T_t^q (u_0, \ell_0) \|_{X_q} + \| T_t^q (u_0, \ell_0) \|_{X_q} \\
  \lesssim C_\tau t^{-N/2} \| (u_0, \ell_0) \|_{X_q} \quad (t \in (0, \tau))
  \]

Similarly,
  \[
  \| u(t) \|_{N+2,q} + |\ell(t)| \lesssim C_\tau t^{-(N+2)/2} \| (u_0, \ell_0) \|_{X_q}
  \]

Sobolev embedding and interpolation gives,
  \[
  \| T_t U \|_{X_r} \lesssim C_\tau t^{-3/2(1/q-1/r)} \| U \|_{X_q}, \quad t \in (0, \tau].
  \]
Let $1 < q < \infty$ and $R \gg 1$.

**Theorem**

For $\lambda \in \Sigma_\theta$ and near $0$,

$$|\lambda|^{-1/2} \| (\lambda I - A_q)^{-1} F \|_{q,-2} \lesssim \| F \|_q,$$

for $F = 0$ in $|x| \geq R$ and the norm is $(1 + |x|^2)^{-2} F \in L^q$.

\[\Downarrow\]

$$\| \mathbb{T}_t U \|_{q,B(0,R)} \leq C(q,R)(1 + t)^{-3/2} \| U \|_q,$$

for $U = 0$ for $|x| > R$. 
Decay estimates of the linear FSI

Step 1. Extension of initial data: Let $t \geq 1$ and $U_0 \in \text{Ran} \mathbb{T}_1 \sim \mathcal{D}(\mathbb{A}_q^m)$. Let $\psi$ be a function in $\mathbb{R}^3$ such that

$$\psi = u_0 \text{ in } E, \quad \text{div}\psi = 0, \quad \|\psi\|_{2m,q} \lesssim \|U_0\|_{\mathcal{D}(\mathbb{A}_q^m)}.$$ 

Step 2. Stokes system in $\mathbb{R}^3$: We consider

$$\partial_t v_0 - \Delta v_0 = 0, \quad \text{div}\, v_0 = 0, \quad v_0(0) = \psi$$

we have

$$\|\nabla^m v_0(t)\|_r \lesssim (1 + t)^{-3/2(1/q-1/r)-m/2}$$
Step 3. Bogoviskii correction: Let $\phi$ be a cut off functions $\phi = 1$ in $|x| \leq R - 2$, and $\phi = 0$ in $|x| > R - 1$. Let $v_1$ is such that

$$\text{div} v_1 = -\nabla \phi \cdot v_0, \quad \text{supp} v_1 \subset \{ R - 2 \leq |x| \leq R - 1 \}.$$ 

with

$$\|v_1(t)\|_{m,q} \lesssim (1 + t)^{-3/2q}.$$ 

Step 4. Local decay estimates: We set $v_2 = u - (1 - \phi)v_0 + v_1$. Then $v_2, \pi, \ell$ and $\omega$ satisfy

$$\begin{cases}
\partial_t v_2 - \mu \Delta v_2 + \nabla \pi = f, & \text{div} v_2 = 0 \quad (t > 0, \ x \in E), \\
v_2 = \ell + \omega \times x & (t > 0, x \in \partial O), \\
m \dot{\ell} + \int_{\partial O} \sigma(v_2, \pi) \nu \, ds = 0 & (t > 0), \\
\mathcal{J} \dot{\omega} + \int_{\partial O} x \times \sigma(v_2, \pi) \nu \, ds = 0 & (t > 0), \\
v_2(0, x) = \zeta(x) & (x \in E), \\
\ell(0) = \ell_0, \quad \omega(0) = \omega_0,
\end{cases}$$
with $f$ and $\zeta$ depending only on $v_0, v_1$ and compactly supported. This gives
\[
\|v_2(t, \cdot)\|_{2m,q,B_R} + |\ell(t)| + |\omega(t)| \lesssim (1 + t)^{-3/2q}.
\]

**Step 5.** Estimate near spatial infinity: We take a cut off function $\chi = 1$ in $|x| \geq R$ and $\chi = 0$ in $|x| \leq R - 1$. We take $v_3$ such that
\[
\text{div} v_3 = \text{div} \chi u.
\]

Next, we define $v_4 = \chi u - v_3$. And, $v_4$ solves stokes system in $R^3$.
\[
\|u(t)\|_{r, |x| \geq R} \lesssim (1 + t)^{-3/2(1/q-1/r)}.
\]
Proof of the main result.

We recall

\[ V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s} F(V(s)), \quad F = (v - \ell) \cdot \nabla v. \]

**Step 1.** Existence and uniqueness in \( \mathbb{X}_3 \)– Fixed point argument: We consider the space

\[ C_T = \left\{ t^{1/4} V \in C([0, T]; \mathbb{X}^6), \quad t^{1/2} V \in C([0, T]; \mathbb{X}^\infty) \right\} \]

\[ \min\{1, t^{1/2}\} \nabla v \in C([0, T]; L^3) \}

If \( \|V_0\|_{\mathbb{X}_3} \) is small then, then we have existence and uniqueness in \( C_\infty \), together with

\[ \|V(t)\|_{\mathbb{X}_p} \lesssim t^{-3/2(1/3-1/p)}, \quad p \in [3, \infty]. \]

This is not enough to conclude large time behaviour of the rigid body.
Step 2: Improved decay estimates: Take $1 < q < 3/2$ and $V_0 \in X_3 \cap X_q$. Then

$$\|T_t V_0\|_\infty \lesssim t^{-3/2q} \|V_0\|_q$$

and

$$\left\| \int_0^t T_{t-s} F(V(s)) \right\|_\infty \lesssim t^{-3/2q} (\|V_0\|_q + \|v_0\|_3).$$
Open questions

- determining the limit from initial data.
- Non-linear problem for rigid body of arbitrary shape.
- Control problem for unbounded domain.....
Happy Birthday Marius.