Large time behaviour of a fluid-structure interaction model

Debayan Maity

TIFR Centre for Applicable Mathematics

Control and Analysis of PDE systems, in honour of Marius Tucsnak for his 60th birthday, Bordeaux

Joint works with Sylvain Ervedoza and Marius Tucsnak.

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Outline

1. 1D FSI models.
2. 3D incompressible NS + Rigid body

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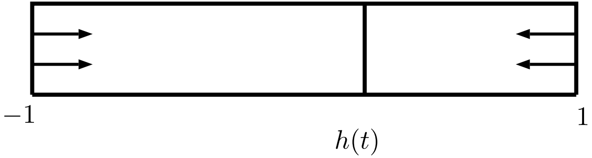
1D simplified model

- We consider a one dimensional model for the motion of a particle (piston) in a cylinder filled with a viscous fluid.



- fluid-piston system evolves in the interval $(-1, 1)$ or $\mathbb{R}$ and $h : [0, \infty) \mapsto (-1, 1)($ or $\mathbb{R})$ denotes the position of the particle.
- The extremities of the cylinder are fixed in the case of BVP.
- The fluid is modelled by the 1D viscous Burgers equation, whereas the piston obeys Newton's second law.

Governing equations

Motion of the gas:
Described in the Eulerian coordinate system by its velocity $u = u(t, x)$, which satisfy the one dimensional Burgers equation

$$\partial_t u + u\partial_x u - \partial_{xx} u = 0, \qquad t \geqslant 0, x \neq h(t)$$
$$(BVP) \sim u(-1, t) = 0 = u(1, t)$$

Motion of the Piston: Described by Newtons law :

$$m\ddot{h}(t) = [\partial_x u](t, h(t)) \qquad (t \geqslant 0),$$

where m is the mass of the piston and the symbol $[f](t, x)$ stands for the jump at instant t of f at x, i.e.,

$$[f](t, x) = f(t, x^+) - f(t, x^-).$$

Equality of the velocity:

$$u(t, h(t)) = \dot{h}(t) \qquad (t \geqslant 0),$$

The position of the piston (and, consequently, the domain occupied by the gas) is one of the unknowns of the problem, we have a free boundary value problem.

Goal

- Existence and uniqueness of solutions.
- Large time behaviour of the point particle:
  - $h(t)$ remains bounded for all time.
  - Particle escapes to spatial infinity.
  - BVP $\sim$ (No) contact with the fluid boundary.
- Control and Stabilization problem: move the point particle from one point to another.

Well-posedness: Vázquez and Zuazua (03, 05),

Control and Stabilization : Doubova and Fernández-Cara (05), Liu, Takahashi and Tucsnak (13), Cîndea et. al. (15), Ramaswamy, Roy and Takahashi (20), Badra and Takahashi (14)

Cauchy problem

- For any $(u_0, \ell_0) \in L^2(\mathbb{R}) \times \mathbb{R}$, the system admits a unique global solution $u \in C([0, \infty); L^2(\mathbb{R}))$, $h \in C^1([0, \infty))$.
- $u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $M = \int_{\mathbb{R}\setminus\{0\}} u_0 dx + m\ell_0 \neq 0$, then

$$\|u(t)\|_{L^r(\mathbb{R})} \lesssim t^{-1/2(1-1/r)}, \qquad t > 0,\ r \geqslant 1.$$

$$\frac{1}{c}(1+t)^{-1/2} < h'(t) < c(1+t)^{-1/2},$$

  for some $c \geqslant 1$, and c is explicit in terms of M.
- Thus the particle escapes to spatial infinity: $\int_0^\infty (1+t)^{-1/2} = \infty$.

BVP

- For any $h_0 \in (-1,1)$, $(u_0, \ell_0) \in H_0^1(-1,1)$ with $u_0(h_0) = \ell_0$ the system admits a unique global strong solution

$$u \in L^2(0,T;H^2) \cap H^1(0,T;L^2), \qquad h \in H^2(0,T),$$

$$\text{(No contact) } h(t) \in (-1,1) \text{ for all } t \geqslant 0.$$

- We also have

$$\|u(t)\|_{L^2} + |h'(t)| \lesssim e^{-\gamma t} \text{ and } \lim_{t\to\infty} h(t) = h_* \in (-1,1).$$

- What is h_*?

1D piston problem

- Fluid: Compressible Navier-Stokes.
- ρ : density of the fluid, u : velocity of the fluid.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & t > 0, x \in \mathbb{R} \setminus \{h(t)\}, \\ \rho(\partial_t u + u\partial_x u) - \partial_{xx} u + \partial_x(\rho^\gamma) = 0 & t > 0, x \in \mathbb{R} \setminus \{h(t)\}, \\ u(t, h(t) \pm 0) = h'(t) & t > 0, \\ mh''(t) = [\partial_x u - \rho^\gamma](t, h(t)) & t > 0, \\ + \text{ initial data} & \end{cases}$$

- Bounded domain - fluid domain: $(-1, 1) \setminus \{h(t)\}$ and $u(t, \pm 1) = 0$.

1D piston problem

- Unbounded domain : Koike (2020)

$$|h'(t)| \lesssim (1+t)^{-3/2} \implies |h(t)-h_0| \leqslant C.$$

- Bounded domain : Shelukhin(1977, 1982)

$$\|u(t)\|_2 \to 0, \quad \|\rho(t)-\overline{\rho}\|_2 \to 0, \quad h'(t) \to 0$$

$$\lim_{t\to\infty} h(t) = \frac{M_L - M_R}{M_L + M_R}, \qquad M_L = \int_{-1}^{h_0} \rho_0, \; M_L = \int_{h_0}^{1} \rho_0,$$

- Adiabatic piston in bounded domain: Feireisl et. al. (18) $\sim \lim_{t\to\infty} h(t) = h_*$, and h_* is unknown.

Our result

Let us consider fluid rigid body problem in whole $\mathbb{R}^3$, where fluid is incompressible Navier-Stokes and the rigid body is a ball.

Theorem

If the initial datum is small enough in suitable norm then the position of the center of the rigid ball converges to some $h_* \in \mathbb{R}^3$ as time goes to infinity.

$\sim$ S. Ervedoza, D. Maity and M. Tucsnak, Large time behaviour for the motion of a solid in a viscous incompressible fluid, (hal-02545798).

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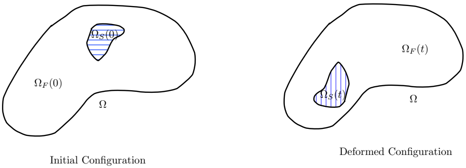
3D problem



- Fluid + rigid body $= \Omega \subset \mathbb{R}^3$ or Fluid + rigid body $= \mathbb{R}^3$.
- $\Omega_s(t) \sim$ the domain occupied by the solid at time $t \geqslant 0$.
- $\operatorname{dist}(\Omega_s(0), \partial\Omega) \geqslant \nu > 0$.
- $h(t) \in \mathbb{R}^3 \sim$ the position of the center of mass,
  $Q(t) \in SO_3(\mathbb{R}) \sim$ the orthogonal matrix giving the orientation of the solid,
  $\omega(t) \in \mathbb{R}^3 \sim$ the angular velocity of the rigid body.
- $\Omega_s(t) = h(t) + Q(t)y$, $y \in \Omega_s(0)$,
- Fluid domain: $\Omega_f(t) = \Omega \setminus \Omega_s(t)$ or $\Omega_f(t) = \mathbb{R}^3 \setminus \Omega_s(t)$

Governing equations:

- Fluid equation: Incompressible Navier-Stokes-Fourier equations:

For $(t,x) \in (0,\infty) \times \Omega_f(t)$,

$$
\begin{aligned}
&\rho_f(\partial_t u + u \cdot \nabla u) - \mathrm{div}\sigma(u,\pi) = 0, \quad \mathrm{div}\ u = 0,\\
&\sigma(u,\pi) = \nu(\nabla u + \nabla u^\top) - \pi I_3,\\
&u(t,x) = h'(t) + \omega(t) \times (x - h(t)) \quad x \in \partial\Omega_s(t),\\
&mh''(t) = -\int_{\partial\Omega_s(t)} \sigma(u,p)n \,\mathrm{d}\gamma,\\
&J\omega'(t) = (J\omega) \times \omega - \int_{\partial\Omega_s(t)} (x - h(t)) \times \sigma(u,p)n \,\mathrm{d}\gamma,\\
&+ \text{ initial data}
\end{aligned}
$$

In bounded domain $u = 0$ on $\partial\Omega$.

Existence and uniqueness

$\sim$ Serre(1987), Takahashi (02), Feireisl (03), Takahashi and Tucsnak(04), Cumsille and Takahashi (08), Geissert et. al (13) and many more..

- Global existence of weak solutions.
- Global existence of strong solution in 2D.
- Local in time or global in time for small data in the 3D case.
- Similar results for bounded domain.

Large time behaviour : bounded domain

$\sim$ Takahashi (04), Maity and Tucsnak (18)

- For initial data sufficiently small

$$\|u(t)\|_{L^q} + |h'(t)| + |\omega(t)| \lesssim e^{-\eta t},$$

  for some $\eta > 0$.

- $q \in (4/3, \infty)$ in $2D$ and $q \in (5/3, \infty)$ in $3D$.
- $h(t)$ remains away from the boundary, $\lim_{t\to\infty} h(t) = h_*$.
- We don't know what is h_*.

Large time behaviour:2D unbounded domain

$\sim$ Ervedoza, Hillairet and Lacave (2014)

- Rigid body is a ball.
- For initial data sufficiently small

$$\|u(t)\|_{L^2} \to 0, \qquad |h'(t)| \lesssim \frac{1}{t}.$$

- Possible unbounded trajectory for the rigid ball. In fact, for the linearized problem, both bounded and unbounded trajectories are possible for the rigid ball, depending on the "mass:" $(m-\pi)\ell_0$.

Our result: 3D unbounded domain and rigid body is a ball

Theorem

If the initial datum is small enough in suitable norm then the position of the center of the rigid ball converges to some $h_* \in \mathbb{R}^3$ *as time goes to infinity.*

$\sim$ Dimension plays a role. Heuristically, L^∞ norm of velocity, for initial data in L^q, decays like $\sim t^{-n/2q}$.

Why we need rigid body to be a ball

- Reformulate the problem in a fixed domain, decay estimates for the linearized problem and a suitable fixed point.
- Use the change of variable $x \to Q(t)x + h(t)$. More precisely,

$$v(t) = Q(t)^\top u(t, Q(t)x + h(t)),$$
$$\ell(t) = Q(t)^\top h(t), \omega(t) = Q(t)^\top \omega(t)$$

Transformed system: in $(0,\infty)\times\Omega_f(0)$, $\Omega_f(0)=\mathbb{R}^3\setminus\Omega_s(0)$:

$$
\begin{aligned}
&\partial_t v-\mathrm{div}\sigma(v,\pi)=-(v-\ell)\cdot\nabla v+(\omega\times y)\cdot\nabla v,\quad \mathrm{div}\ v=0,\\
&v(t,y)=\ell(t)+\omega(t)\times y,\quad y\in\partial\Omega_s(0),\\
&m\ell'(t)=-\int_{\partial\Omega_s(0)}\sigma(v,\pi)n\,\mathrm{d}\gamma,\\
&J\omega'(t)=-\int_{\partial\Omega_s(0)}y\times\sigma(v,\pi)n\,\mathrm{d}\gamma,\\
&\ +\ \text{initial data}
\end{aligned}
$$

$\sim$ if the rigid body is not a ball, simple change of variable gives the spatial unbounded term : $(\omega\times y)\cdot\nabla u$

$\sim$ We need more complicated change of variable, which induces a lot of nonlinear term.

Mild solutions

We set

$$V = \begin{cases} v & \text{in } \mathbb{R}^3 \setminus B(0,1) \\ \ell + \omega \times y & \text{in } B(0,1). \end{cases}$$

and, for $1 < q < \infty$,

$$\mathbb{X}^q = \left\{ \Phi \in L^q(\mathbb{R}^3)^3 \mid \mathrm{div}\Phi = 0 \text{ in } \mathbb{R}^3, D(\Phi) = 0 \text{ in } B(0,1) \right\}$$

Then we can rewrite the system

$$V'(t) = \mathbb{A}_q V(t) + F(V), \quad V(0) = V_0,$$

where $\mathbb{A}_q : \mathcal{D}(\mathbb{A}_q) \to \mathbb{X}^q$ is the linear fluid-structure operator and F nonlinear terms.

$$V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s} F(V(s))$$

Our result

- Existence and uniqueness: For $v_0 \in L^3$, $\operatorname{div}\, v_0 = 0$ satisfying compatibility conditions and

$$\|v_0\|_{L^3} + |\ell_0| + |\omega_0| << 1,$$

the system admits a unique solution in $C([0,\infty); L^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$.

- Decay estimates: Let $q \in (1, 3/2)$, $v_0 \in L^q \cap L^3$ with compatibility conditions and same smallness assumption. Then

$$\|u(t)\|_{L^\infty} + |h'(t)| + |\omega(t)| \lesssim t^{-3/2q}$$

$\Longrightarrow$ $h(t)$ converges to h_*. We don't know what is h_*.

Linear FSI system with arbitrary rigid body

$\sim E$ - Fluid domain, $\mathcal{O}-$ rigid body.

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\pi = 0, \quad \mathrm{div} u = 0 & (t \geqslant 0, x \in E), \\ u = \ell + \omega \times x & (t \geqslant 0, x \in \partial\mathcal{O}), \\ m\dot{\ell} + \displaystyle\int_{\partial\mathcal{O}} \sigma(u,\pi)\nu \,\mathrm{d}s = 0 & (t \geqslant 0), \\ \mathcal{J}\dot{\omega} + \displaystyle\int_{\partial\mathcal{O}} x \times \sigma(u,\pi)\nu \,\mathrm{d}s = 0, & (t \geqslant 0), \\ u(0) = u_0 & (x \in E), \\ \ell(0) = \ell_0, \qquad \omega(0) = \omega_0. \end{cases}$$

We rewrite the system as

$$\frac{d}{dt} U = \mathbb{A}_q U, \quad U(0) = U_0.$$

Decay estimates for FSI semigroup

Theorem (SE, DM, MT)

- $1 < q \leqslant r < \infty$ *or* $1 < q < \infty, r = \infty$. *Then*

$$\|\mathbb{T}_t U\|_r \leqslant C(q,r) t^{-3/2(1/q-1/r)} \|U\|_q \quad t > 0, U \in \mathbb{X}^q.$$

- $1 < q \leqslant r \leqslant 3$. *Then*

$$\|\nabla \mathbb{T}_t U\|_{r,E} \leqslant C(q,r) t^{-3/2(1/q-1/r)-1/2} \|U\|_q \quad t > 0, U \in \mathbb{X}^q.$$

$\sim$ The decay estimates are same as Stokes system in the exterior domain (Iwashita, 1989).
$\sim$ Tools: Two types of resolvent estimates and some cut-off arguments.

1st Resolvent estimate

Theorem

Let $1 < q < \infty$. *Then*

$$\|\lambda(\lambda\mathbb{I} - \mathbb{A}_q)^{-1}\| \leqslant C, \text{ for } \lambda \in \Sigma_\theta, \theta > \pi/2.$$

- For fluid (Stokes): Borchers and Sohr (1987).
- $\mathbb{A}_2$ is self adjoint $\implies$ $\mathbb{A}_2$ is sectorial. (Takahashi and Tucsnak, 2004).
- $\mathbb{A}$ is sectorial on $\mathbb{X}^2 \cap \mathbb{X}^q$, $q \geqslant 6$. (Wang and Xin, 2011).
- in bounded domain, $\mathbb{A}_q$ is sectorial for any $1 < q < \infty$. (Maity and Tucsnak, 18).
- Idea : Resolvent of FSI $=$ Resolvent of Stokes $+$ "other terms"

Small time decay estimates

- Consequence of the resolvent estimate:

$$\|\mathbb{A}_q^m \mathbb{T}_t\| \lesssim t^{-m} \text{ for any } m \in \mathbb{N}.$$

- A priori estimate :
$\|u\|_{2m,q} + |\ell| + |\omega| \lesssim \|\mathbb{A}_q^m(u,\ell,\omega)\|_{\mathbb{X}_q} + \|(u,\ell,\omega)\|_{\mathbb{X}_q}$
- $N = 3[1/q - 1/r]$. Assume N is even. Then

$$\begin{aligned} \|u(t)\|_{N,q} + |\ell(t)| + |\omega(t)| &\lesssim \|\mathbb{A}_q^{N/2}\mathbb{T}_t^q(u_0,\ell_0)\|_{\mathbb{X}_q} + \|\mathbb{T}_t^q(u_0,\ell_0)\|_{\mathbb{X}_q} \\ &\lesssim C_\tau t^{-N/2}\,\|(u_0,\ell_0)\|_{\mathbb{X}_q} \quad (t \in (0,\tau)) \end{aligned}$$

Similarly,

$$\|u(t)\|_{N+2,q} + |\ell(t)| \lesssim C_\tau t^{-(N+2)/2}\,\|(u_0,\ell_0)\|_{\mathbb{X}_q}$$

Sobolev embedding and interpolation gives,

$$\|\mathbb{T}_t U\|_{\mathbb{X}_r} \lesssim C_\tau t^{-3/2(1/q-1/r)}\|U\|_{\mathbb{X}_q}, \qquad t \in (0,\tau].$$

2nd resolvent estimate and local decay estimates

Let $1 < q < \infty$ and $R >> 1$.

Theorem

For $\lambda \in \Sigma_\theta$ and near 0,

$$|\lambda|^{-1/2} \|(\lambda I - \mathbb{A}_q)^{-1} F\|_{q,-2} \lesssim \|F\|_q,$$

for $F = 0$ in $|x| \geqslant R$ and the norm is $(1 + |x|^2)^{-2} F \in L^q$.

$$\Downarrow$$

$$\|\mathbb{T}_t U\|_{q,B(0,R)} \leqslant C(q,R)(1+t)^{-3/2} \|U\|_q,$$

for $U = 0$ for $|x| > R$.

Decay estimates of the linear FSI

Step 1. Extension of initial data: Let $t \geqslant 1$ and $U_0 \in \mathrm{Ran}\mathbb{T}_1 \sim \mathcal{D}(\mathbb{A}_q^m)$. Let ψ be a function in $\mathbb{R}^3$ such that

$$\psi = u_0 \text{ in } E,\ \mathrm{div}\psi = 0, \quad \|\psi\|_{2m,q} \lesssim \|U_0\|_{\mathcal{D}(\mathbb{A}_q^m)}.$$

Step 2. Stokes system in $\mathbb{R}^3$: We consider

$$\partial_t v_0 - \Delta v_0 = 0, \quad \mathrm{div} v_0 = 0, \quad v_0(0) = \psi$$

we have

$$\|\nabla^m v_0(t)\|_r \lesssim (1+t)^{-3/2(1/q-1/r)-m/2}$$

Step 3. Bogoviskii correction: Let ϕ be a cut off functions $\phi = 1$ in $|x| \leqslant R-2$, and $\phi = 0$ in $|x| > R-1$.Let v_1 is such that

$$\mathrm{div} v_1 = -\nabla\phi \cdot v_0, \quad \mathrm{supp} v_1 \subset \{R-2 \leqslant |x| \leqslant R-1\}.$$

with

$$\|v_1(t)\|_{m,q} \lesssim (1+t)^{-3/2q}.$$

Step 4. Local decay estimates: We set $v_2 = u - (1-\phi)v_0 + v_1$. Then v_2, π, ℓ and ω satisfy

$$\begin{cases} \partial_t v_2 - \mu\Delta v_2 + \nabla\pi = f, \quad \mathrm{div}\ v_2 = 0 & (t>0,\ x \in E), \\ v_2 = \ell + \omega \times x & (t>0, x \in \partial\mathcal{O}), \\ m\dot{\ell} + \displaystyle\int_{\partial\mathcal{O}} \sigma(v_2,\pi)\nu \, \mathrm{d}s = 0 & (t>0), \\ \mathcal{J}\dot{\omega} + \displaystyle\int_{\partial\mathcal{O}} x \times \sigma(v_2,\pi)\nu \, \mathrm{d}s = 0 & (t>0), \\ v_2(0,x) = \zeta(x) & (x \in E), \\ \ell(0) = \ell_0, \qquad \omega(0) = \omega_0, \end{cases}$$

with f and ζ depending only on v_0, v_1 and compactly supported. This gives

$$\|v_2(t,\cdot)\|_{2m,q,B_R} + |\ell(t)| + |\omega(t)| \lesssim (1+t)^{-3/2q}.$$

Step 5. Estimate near spatial infinity: We take a cut off function $\chi = 1$ in $|x| \geqslant R$ and $\chi = 0$ in $|x| \leqslant R-1$. We take v_3 such that

$$\mathrm{div} v_3 = \mathrm{div} \chi u.$$

Next, we define $v_4 = \chi u - v_3$. And, v_4 solves stokes system in R^3.

$$\|u(t)\|_{r,|x|\geqslant R} \lesssim (1+t)^{-3/2(1/q-1/r)}.$$

Proof of the main result.

We recall

$$V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s} F(V(s)), \quad F = (v - \ell) \cdot \nabla v.$$

Step 1. Existence and uniqueness in $\mathbb{X}_3$– Fixed point argument: We consider the space

$$\mathcal{C}_T = \Big\{ t^{1/4} V \in C([0,T]; \mathbb{X}^6), \quad t^{1/2} V \in C([0,T]; \mathbb{X}^\infty) \\ \min\{1, t^{1/2}\} \nabla v \in C([0,T]; L^3) \Big\}$$

If $\|V_0\|_{\mathbb{X}_3}$ is small then, then we have existence and uniqueness in $\mathcal{C}_\infty$, together with

$$\|V(t)\|_{\mathbb{X}_p} \lesssim t^{-3/2(1/3-1/p)} \qquad p \in [3, \infty].$$

This is not enough to conclude large time behaviour of the rigid body.

Large time behaviour

Step 2: Improved decay estimates: Take $1 < q < 3/2$ and $V_0 \in \mathbb{X}_3 \cap \mathbb{X}_q$. Then

$$\|\mathbb{T}_t V_0\|_\infty \lesssim t^{-3/2q} \|V_0\|_q$$

and

$$\left\| \int_0^t \mathbb{T}_{t-s} F(V(s)) \right\|_\infty \lesssim t^{-3/2q} (\|V_0\|_q + \|v_0\|_3).$$

Open questions

- determining the limit from initial data.
- Non-linear problem for rigid body of arbitrary shape.
- Control problem for unbounded domain.....

Happy Birthday Marius.