Control and Analysis of PDEs

In honour of Marius Tucsnak

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Remarks on the Convergence to equilibrium in some parabolic evolution equations

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- Results concerning non local parabolic equations are obtained in collaboration with Matthieu Alfaro (université de Rouen) and Pierre Gabriel Université de Paris–Saclay (site de Versailles), a work in progress:

  Confining Integro-Differential Equations Originating from Evolutionary Biology: Ground States and Long Time Dynamics.

Today's talk

Introduction

- Consider a Banach space X, a subspace of $L^1(d\mu)$, the space of integrable functions on a measurable set Ω with respect to a positive measure $d\mu$.
- Recall that a linear C_0 semigroup on X is a family of bounded operators $S(t) : X \longrightarrow X$, for $t \geq 0$, such that
$$S(0) = I, \qquad S(t+\tau) = S(t)S(\tau), \qquad \forall u \in X, \quad \lim_{t\to 0} S(t)u = u.$$
- Let $(S(t))_{t\geq 0}$ be a linear C_0 semigroup acting on X which preserves positivity, that is
$$f \in X, \quad f \geq 0 \quad \Longrightarrow \quad S(t)f \geq 0.$$
- Our aim is to study the limit of $S(t)f$ as $t \to +\infty$, for a given $f \geq 0$.

Introduction

- The family under study is a non local diffusion equation of the form

$$(1)\quad \begin{cases} \partial_t f = \sigma^2 (J * f - f) - (W(x) - \overline{W}(f)) f & \text{in } (0,\infty)\times\mathbb{R}^N \\ f(0,x) = f_0(x) \geq 0 \quad \text{in } \mathbb{R}^N, & \displaystyle\int_{\mathbb{R}^N} f_0(x) = 1. \end{cases}$$

where $J \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $W \geq 0$ are such that

$$J \geq 0, \quad \int_{\mathbb{R}^N} J(x)dx = 1, \quad J(-x) = J(x), \qquad \lim_{|x|\to\infty} W(x) = +\infty,$$

while $\overline{W}(u)$ is a nonlocal term defined by

$$\overline{W}(f) := \overline{W}(f(t)) := \langle f(t,\cdot), W \rangle := \int_{\mathbb{R}^N} W(y) f(t,y) dy.$$

This equation is considered in the study of replication-mutation mathematical models in biology.

Introduction

- A classical approach is to study the spectrum $\sigma(L)$ of the generator L of the semigroup, that is the operator L such that $S(t) = \exp(tL)$.

- For instance, one tries to show that $0 \in \sigma(L)$ and that there exists a non trivial $\varphi \geq 0$ verifying $L\varphi = 0$.

- Then one tries to prove that L has a spectral gap: this means that there exists $\omega_0 > 0$ verifying
$$\sigma(L) \setminus \{0\} \subset \{z \in \mathbb{C}\,;\, \mathrm{Re}(z) < -\omega_0\}.$$

- Finally one shows that $S(t)f_0$ converges to a multiple of φ provided $f_0 \geq 0$, with a decay rate of order e^{-at} for some a verifying $0 < a \leq \omega_0$. ■

Döblin's theorem

A very elegant result is due to Wolfgang Döblin (1915 — 1940), sent to the French « Académie des Sciences » in February 1940 in a sealed document (number 11.688), which was unsealed and made public only in 2000.

- Consider a positive semigroup $(S(t))_{t\geq 0}$ acting on the space $L^1(d\mu)$ and such that for $f \in L^1(d\mu)$ we have

  $$M(S(t)f) = M(f),$$

  where we denote by $M(f)$ the total mass of f, that is

  $$M(f) := \int f d\mu.$$

  Then we have

Döblin's theorem

1 Döblin's Theorem. *Assume that there exists* $T > 0$, *a number* $\theta \in (0,1)$, *and a function (or measure…)* $\varphi \geq 0$ *for which* $M(\varphi) = 1$ *and such that for all* $g \in L^1(d\mu)$, *verifying* $g \geq 0$, *one has*

$$S(T)g \geq \theta\, M(g)\, \varphi. \tag{2}$$

Then for any $f \in L^1(d\mu)$ *such that* $M(f) = 0$ *one has*

$$\|S(T)f\|_{L^1(d\mu)} \leq (1-\theta)\, \|f\|_{L^1(d\mu)}\,, \tag{3}$$

and also

$$\|S(t)f\|_{L^1(d\mu)} \leq \mathrm{e}^{-\omega t}\, \|f\|_{L^1(d\mu)}\,, \tag{4}$$

with

$$\omega := \frac{-\log(1-\theta)}{T} > 0.$$

Döblin's theorem

- Indeed, if $M(f) = 0$, write $f = f^+ - f^-$ with $f^\pm := \max(0, \pm f)$, and set
$$\lambda := M(f^+) = \int f^+ d\mu = \int f^- d\mu.$$
- Then
$$\begin{aligned} \|S(T)f\|_{L^1(d\mu)} &= \|(S(T)f^+ - \theta\lambda\varphi) + (\theta\lambda\varphi - S(T)f^-)\|_{L^1(d\mu)} \\ &\le \|S(T)f^+ - \theta\lambda\varphi\|_{L^1(d\mu)} + \|\theta\lambda\varphi - S(T)f^-\|_{L^1(d\mu)}. \end{aligned} \tag{5}$$
- But thanks to **(2)** we have $S(T)f^\pm - \theta\lambda\varphi \ge 0$, and so
$$\|S(T)f^\pm - \theta\lambda\varphi\|_{L^1(d\mu)} = \int (S(T)f^\pm - \theta\lambda\varphi)\, d\mu = (1-\theta)\lambda,$$
which, plugged into **(5)** and noting that $2\lambda = \|f\|_{L^1(d\mu)}$, yields
$$\|S(T)f\|_{L^1(d\mu)} \le 2(1-\theta)\lambda = (1-\theta)\,\|f\|_{L^1(d\mu)}.$$

Döblin's theorem

- Now the proof of the exponential decay of $S(t)f$ when $M(f) = 0$, is straightforward: indeed since $|S(t)f| \leq S(t)|f|$, we have
$$\|S(t)f\|_{L^1(d\mu)} \leq \|S(t)|f|\|_{L^1(d\mu)} = \|f\|_{L^1(d\mu)},$$
and so $S(t)$ is a contractive semigroup in $L^1(d\mu)$.

- If $t > 0$, write $t = nT + \tau$ for $\tau \in [0, T)$ and $n \geq 0$ integer. Noting that
$$c_1 := \sup_{0 \leq \sigma \leq T} \|S(\sigma)\| = 1,$$
and using the semigroup property of $S(t)$ we get
$$\|S(t)f\|_{L^1(d\mu)} \leq c_1 \|S(T)^n f\|_{L^1(d\mu)} \leq (1-\theta)^n \|f\|_{L^1(d\mu)}.$$
Finally, since $n = (t-\tau)/T$ we deduce (4). ∎

Döblin's theorem

- In particular, if we know that

$$\forall t \geq 0, \quad S(t)\varphi = \varphi, \qquad M(\varphi) = 1, \qquad \varphi \geq 0,$$

then applying the above result to the function

$$f - M(f)\varphi,$$

we conclude that for any initial data $f \geq 0$ we have

$$\|S(t)f - M(f)\varphi\|_{L^1(d\mu)} \leq \mathrm{e}^{-\omega t} \, \|f - M(f)\varphi\|_{L^1(d\mu)}.$$

- Despite this result being very elegant and powerful, the main issue is to prove **(2)**, even if one succeeds to prove the existence of the ground state φ. Indeed, in most interesting cases, one has $\varphi > 0$ in $\mathbb{R}^N$ and it is not easy to show a uniform lower bound for $S(t)f$ when $f \geq 0$ and $f \not\equiv 0$. ■

Nonlocal equation: ground state

- We study existence of a positive stationary solution (or ground state) for the equation

$$
(6) \qquad \begin{cases} \partial_t f = \sigma^2 (J * f - f) - (W(x) - \overline{W}(f))f & \text{in } (0,\infty) \times \mathbb{R}^N \\ f(0,x) = f_0(x) \geq 0 \text{ in } \mathbb{R}^N, & M(f_0) = 1. \end{cases}
$$

- This amounts to find $f_0 \geq 0$ such that $f_0 \not\equiv 0$ and

$$\sigma^2 (J * f_0 - f_0) - W(x) f_0 + \overline{W}(f_0) f_0 = 0,$$

so that setting $\lambda := \overline{W}(f_0) > 0$, we end up looking for $f_0 \geq 0$ solution to

$$-\sigma^2 J * f_0 + W f_0 = (\lambda - \sigma^2) f_0.$$

Nonlocal equation: ground state

- This is an eigenvalue problem: if $\varphi \geq 0$ is an eigenfunction and $\lambda_1 \in \mathbb{R}$ are such that

$$(7) \qquad -\sigma^2 J * \varphi + W\varphi = \lambda_1 \varphi, \qquad \int_{\mathbb{R}^N} |\varphi(x)|^2 \, dx = 1,$$

then setting

$$\lambda := \lambda_1 + \sigma^2, \qquad f_0 := \alpha\varphi, \quad \text{with} \quad \alpha := \left(\int_{\mathbb{R}^N} \varphi(x) W(x) dx \right)^{-1} \lambda,$$

we have a stationary solution for **(6)**, and $M(f_0) = 1$ (actually one has also $\lambda = \lambda_1 + \sigma^2 > 0$ so that $f_0 \geq 0$).

- Hence we are going to prove that the linear operator associated to equation **(6)** has a positive eigenfunction, associated to an eigenvalue $\lambda_1 > -\sigma^2$.

Nonlocal equation: ground state

- Assume that $J \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ satisfies

(8) $$J \geq 0, \quad J(x) = J(-x), \quad \int_{\mathbb{R}^N} J(x)dx = 1,$$

and

(9) $$\exists J_0 > 0,\ \exists r_0 > 0, \quad J(x) \geq J_0 \ \text{ on } \ B(0, r_0).$$

- The potential W satisfies

(10) $$W \in C(\mathbb{R}^N), \qquad W \geq 0, \qquad \lim_{|x|\to\infty} W(x) = +\infty,$$

and

(11) $$\exists x_* \in \mathbb{R}^N,\ \exists r_0 > 0, \quad W(x_*) = 0, \quad \frac{1}{W} 1_{[W>0]} \notin L^1(B(x_*, r_0)).$$

Nonlocal equation: ground state

- Then consider the operator $(L, D(L))$ acting on $L^2(\mathbb{R}^N)$ defined by

$$Lu := -\sigma^2 J * u + Wu, \quad D(L) := L^2(1+W).$$

equation **(6)** can be written as

$$\partial_t f = -Lf - \sigma^2 f + \overline{W}(f)f, \qquad f(0) = f_0 \geq 0, \qquad M(f_0) = 1,$$

and the eigenvalue problem **(7)** reads

$$L\varphi = \lambda_1 \varphi, \qquad \|\varphi\|^2 = 1,$$

and one would like to have $\varphi \geq 0$ and $\lambda_1 + \sigma^2 > 0$.

Nonlocal equation: ground state

- Equation **(1)** has been widely studied, when instead of the linear non local diffusion $\sigma^2(J * f - f)$ one has a diffusion given for instance by $-(-\Delta)^s$ for $0 < s \leq 1$, or in some special cases of J and W (for instance M. Alfaro, J. Coville, R. Bürger, F. Li, X. Wang, G. Legendre).

- For a diffusion equation like
$$\partial_t f = \Delta f - (|x|^4 - |x|^2) f$$
one can show easily that there exists a ground state $\varphi > 0$ and $\lambda_1 \in \mathbb{R}$ such that
$$-\Delta\varphi + (|x|^4 - |x|^2)\varphi = \lambda_1 \varphi.$$
More generally the existence of a ground state for equations such as
$$-\Delta\varphi + W\varphi = \lambda_1 \varphi$$
has been widely studied (see M. Reed & B. Simon, L.A. Takhtajan).

Nonlocal equation: ground state

2 Theorem. *There exists* $\varphi \in C_0(\mathbb{R}^N) \cap L^2(1+W)$ *and* $\lambda_1 \in \mathbb{R}$*, an eigenvalue of multiplicity one, such that*

$$-\sigma^2 J * \varphi + W\varphi = \lambda_1 \varphi,$$

and moreover

$$-\sigma^2 < \lambda_1 < 0, \qquad \forall x \in \mathbb{R}^N, \quad 0 < \varphi(x) \leq \frac{\sigma^2 \|J\|_{L^2(\mathbb{R}^N)}}{W(x) - \lambda_1}.$$

Nonlocal equation: ground state

- Define

$$E(u) := (Lu|u) = -\sigma^2 \int_{\mathbb{R}^N} (J * u)(x)u(x)dx + \int_{\mathbb{R}^N} W(x)u^2(x)dx.$$

and the manifold

$$S := \left\{u \in L^2(1+W) \; ; \; \int_{\mathbb{R}^N} u^2(x)dx = 1\right\}.$$

We want to show that

$$\lambda_1 := \inf_{u \in S} E(u)$$

is achieved for some $\varphi \in S$ such that $\varphi \geq 0$.

- The difficulty here is that we do not have any compactness in the imbedding $L^2(1+W) \subset L^2(\mathbb{R}^N)$.

Nonlocal equation: ground state

- One shows first that there exists $u \in S$ such that $E(u) < 0$ and thus $\boldsymbol{\lambda_1 < 0}$. (This is the crucial step...).

- Then one verifies that if $u_n \rightharpoonup u$ in $L^2(1+W)$ one has

$$\sigma^2 \int_{\mathbb{R}^N} (J * u_n) u_n dx \to \sigma^2 \int_{\mathbb{R}^N} (J * u) u dx.$$

  This is due to the fact that on the one hand for $x \in \mathbb{R}^N$ we have $K(x - \cdot) \in L^2(\mathbb{R}^N)$ and thus $K * u_n(x) \to K * u(x)$, and on the other hand $u_n \in L^2(1+W)$.

- Next observe that $E(|u|) \le E(u)$, so that if

$$\varphi_n \in S \qquad \text{satisfies} \qquad \lambda_1 \le E(\varphi_n) \le \lambda_1 + \frac{1}{n},$$

  we can assume that $\varphi_n \ge 0$.

- Moreover since

$$\int W(x)\varphi_n(x)^2 dx = E(\varphi_n) + \sigma^2 \int (J * \varphi_n)\varphi_n dx \leq \lambda_1 + 1 + \sigma^2,$$

assuming that $\varphi_n \rightharpoonup \varphi$ in $L^2(1+W)$

$$0 \leq \int W(x)\varphi_n(x)^2 dx = E(\varphi_n) + \sigma^2 \int (J * \varphi_n)\varphi_n \to \lambda_1 + \sigma^2 \int (J * \varphi)\varphi dx,$$

and by Fatou's lemma

$$\int W(x)\varphi(x)^2 dx \leq \lambda_1 + \sigma^2 \int (J * \varphi)\varphi dx \leq \lambda_1 + \sigma^2.$$

Since $\boldsymbol{\lambda_1 < 0}$, this implies that $\boldsymbol{\varphi \not\equiv 0}$, and hence $\boldsymbol{\lambda_1 + \sigma^2 > 0}$.

Nonlocal equation: ground state

- Now the same inequality shows that

$$E(\varphi) \leq \lambda_1 \qquad \text{and obviously} \quad \int_{\mathbb{R}^N} \varphi(x)^2 dx \leq 1.$$

Therefore, if $\theta := \|\varphi\|_{L^2} < 1$, upon considering $\tilde{\varphi} := \varphi/\theta$, we would get

$$\tilde{\varphi} \in S, \qquad \lambda_1 \leq E(\tilde{\varphi}) = \theta^{-2} E(\varphi) \leq \theta^{-2}\lambda_1 < \lambda_1,$$

since $\lambda_1 < 0$. This means that we must have $\|\varphi\|_{L^2} = 1$ and $\varphi \in S$ while $E(\varphi) = \lambda_1$.

- Finally since

$$\varphi = \frac{\sigma^2 J * \varphi}{W - \lambda_1} \tag{12}$$

and $J * \varphi \in C_0(\mathbb{R}^N)$, we infer that $\varphi \in C_0(\mathbb{R}^N)$. $\square$

Nonlocal equation: ground state

- In order to prove the positivity of φ, one may use **(12)**, but in order to show the simplicity of λ_1 we need a strong maximum principle for the operator L.

> **3 Lemma.** *Let $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}^N)$. If $f \ge 0$ and $f \not\equiv 0$ is given and $u \in L^p(\mathbb{R}^N)$ solves*
>
> $$Lu + \lambda u = f,$$
>
> *for some $\lambda > \sigma^2$, then $u \ge 0$, and if $u \in C_0(\mathbb{R}^N)$ we have $u > 0$.*

- When $p < \infty$, write

$$\sigma^2 J * u^- + (W + \lambda)u = f + \sigma^2 J * u^+ \ge 0$$

and then multiply by $(u^-)^{p-1} 1_{[u<0]}$ to get

$$\int_{\mathbb{R}^N} \left(W(x) + \lambda - \sigma^2\right) |u^-(x)|^p \, dx \le 0 \quad \Longrightarrow \quad u^- \equiv 0.$$

Nonlocal equation: ground state

- When $p = \infty$, denote
$$m := \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} u(x).$$
Since $J \geq 0$, we have
$$\sigma^2 \int_{\mathbb{R}^N} J(x-y)u(y)dy \geq m\sigma^2 \int J(x-y)dy = m\sigma^2,$$
so that
$$(W(x)+\lambda)u = f + \sigma^2 J * u \geq \sigma^2 m.$$
- If $m < 0$, consider a sequence $(x_n)_n$ such that $u(x_n) \to m$ with $m \leq u(x_{n+1}) \leq u(x_n) < 0$, and
$$0 \leq u(x_n) - m \leq \frac{-1}{2\lambda}(\lambda - \sigma^2)m.$$

Nonlocal equation: ground state

- One would get

$$0 \leq -W(x_n)u(x_n) \leq \lambda(u(x_n) - m) + (\lambda - \sigma^2)m \leq \frac{1}{2}(\lambda - \sigma^2)m < 0.$$

Consequently one must have $m \geq 0$, that is $u \geq 0$ a.e. $\square$

Nonlocal equation: ground state

- In order to prove $\varphi > 0$ note that if $\lambda > \sigma^2$ and $f := (\lambda_1 + \lambda)\varphi$, we have $f \geq 0$ and $f \not\equiv 0$

$$L\varphi + \lambda\varphi = f \geq 0$$

therefore $\varphi > 0$, thanks to the previous lemma.

- Next one shows:

4 Lemma. *The eigenvalue λ_1 is simple.*

Proof.

- Otherwise there would exist $v \not\equiv 0$ such that $v^{\pm} \not\equiv 0$ and $Lv = \lambda_1 v$.
- In particular $\lambda_1 \leq E(|v|) \leq E(v) = \lambda_1$, and thus $L|v| = \lambda_1 |v|$.

Nonlocal equation: ground state

- Therefore $Lv^{\pm} = \lambda_1 v^{\pm}$ and for $\lambda > \sigma^2$ by Lemma 3 we have

  $$Lv^{\pm} + \lambda v^{\pm} = (\lambda_1 + \lambda) v^{\pm} \geq 0 \qquad \Longrightarrow \qquad v^{\pm} > 0 \text{ in } \mathbb{R}^N.$$

  Since $(v^{+} | v^{-}) = 0$, this is a contradiction. ∎

Spectral gap

- The eigenfunction φ being found with a simple eigenvalue $\lambda_1 < 0$, we define

$$S_0 := \left\{ u \in L^2(1+W)\ ;\ (u|\varphi) = 0 \text{ and } \int_{\mathbb{R}^N} |u(x)|^2\, dx = 1 \right\}, \tag{13}$$

and

$$\lambda_2 := \inf_{u \in S_0} E(u). \tag{14}$$

- Then we have

5 Lemma. *We have either $\lambda_2 \geq 0$ or there exists $\psi \in S_0$ such that*

$$\lambda_1 < \lambda_2 = E(\psi) < 0.$$

Spectral gap

- Indeed, if $\lambda_2 < 0$, following the same arguments as in the proof of Theorem 2, one shows that λ_2 is achieved for some $\psi \in S_0$.
- Now, since $\psi \not\equiv 0$, and $(\varphi|\psi) = 0$, we have $\psi^{\pm} \not\equiv 0$, and

$$\lambda_2 = E(\psi) = E(\psi^+) + E(\psi^-) + 2\sigma^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(y)\psi^+(x-y)\psi^-(x)dxdy.$$

- However $E(\psi^{\pm}) > \lambda_1 \|\psi^{\pm}\|^2$, since $\psi^{\pm}$ cannot be a multiple of φ, and

$$E(\psi^+) + E(\psi^-) > \lambda_1 \|\psi^+\|^2 + \lambda_1 \|\psi^-\|^2 = \lambda_1 \|\psi\|^2 = \lambda_1.$$

- Also clearly

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(y)\psi^+(x-y)\psi^-(x)dxdy \geq 0,$$

and so $\lambda_2 > \lambda_1$. ■

Spectral gap in the general case

- Since $(Av|v) \leq 0$, the semigroup $S_A(t) := \mathrm{e}^{tA}$ is contractive in $L^2(\mathbb{R}^N)$
$$\|S_A(t)\|_{L^2 \to L^2} \leq 1.$$
- In what follows let $1 \leq p < \infty$. Recall (cf. the book *One parameter semigroups* by K.J. Engel & R. Nagel, chapter 4, § 2) that the *growth bound* of S_A is defined as being
$$\omega_0(A) := \inf \left\{ \omega \in \mathbb{R}\,;\, \exists M > 0,\ \forall t \geq 0,\ \|S_A(t)\|_{L^p \to L^p} \leq M\mathrm{e}^{\omega t} \right\},$$
and the *spectral bound* of A (which is self-adjoint in $L^2(\mathbb{R}^N)$) is
$$s(A) := \sup \left\{ \mathrm{Re}(\lambda)\,;\, \lambda \in \sigma(A) \right\} = \sup \left\{ \lambda\,;\, \lambda \in \sigma(A) \right\}.$$
- Thus in our case we have $\omega_0(A) = s(A) = 0$.

Spectral gap in the general case

- We wish to show that there exists $a > 0$ such that $\sigma(A) \setminus \{0\} \subset (-\infty, -a]$, or

$$\omega_0(A_{|\varphi^\perp}) = s(A_{|\varphi^\perp}) \leq -a.$$

- To this end we shall use the notion of essential growth bound, defined by

$$\omega_{\mathrm{ess}}(A) := \inf\left\{\omega \in \mathbb{R}\ ;\ \exists M > 0,\ \forall t \geq 0,\ \|S_A(t)\|_{\mathrm{ess}} \leq M\mathrm{e}^{\omega t}\right\},$$

  where the essential norm of a bounded operator B is defined by

$$\|B\|_{\mathrm{ess}} := \inf\left\{\|B - T\|_{L^p \to L^p}\ ;\ T : L^p(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N) \text{ is compact}\right\}.$$

- Clearly $\omega_{\mathrm{ess}}(A) \leq \omega_0(A)$. One says that S_A is **quasi-compact** if $\omega_{\mathrm{ess}}(A) < 0$, and the main interest of this notion is the following result (Corollary 2.11, chapitre 4, § 2 of the above book by Engel & Nagel)

Spectral gap in the general case

6 Theorem. *If $\omega > \omega_{\text{ess}}(A)$ then*

$$\sigma(A) \cap \left\{\lambda \in \mathbb{C}\,;\, \mathrm{Re}(\lambda) \geq \omega\right\}.$$

is at most a finite number of eigenvalues of A, each having a finite multiplicity.

- In our case, A is self-adjoint (in $L^2(\mathbb{R}^N)$) and we show that $\omega_{\text{ess}}(A) < 0$. Then we fix $0 < a < -\omega_{\text{ess}}(A)$ in order to obtain a decay rate of order e^{-at} for $S_A(t)$ in the space $\varphi^{\perp}$.

- To show that $\omega_{\text{ess}}(A) < 0$, we split A as the sum of two operators A_0 and B

$$A = A_0 + B, \qquad A_0 u := \lambda_1 u - W u, \quad D(A_0) := D(L), \qquad Bu := \sigma^2 J * u.$$

Spectral gap in the general case

- We have $S_{A_0}(t)u_0 = \mathrm{e}^{t(\lambda_1 - W)}u_0$ and

$$\|S_{A_0}(t)\|_{L^p\to L^p} \le \mathrm{e}^{\lambda_1 t}, \qquad \|B\|_{L^p\to L^p} \le \sigma^2.$$

- Writing equation **(17)** in two forms

$$\frac{dv}{dt} = A_0 v + Bv \qquad \text{and} \qquad \frac{dv}{dt} = Bv + A_0 v, \qquad \text{with } v(0) = v_0,$$

and then considering the mild solution of each of these forms we obtain

$$\begin{aligned} v(t) &= S_{A_0}(t)v_0 + \int_0^t S_{A_0}(t-\tau)Bv(\tau)d\tau \\ &= S_B(t)v_0 + \int_0^t S_B(t-\tau)A_0 v(\tau)d\tau, \end{aligned}$$

one obtains the Duhamel formula

Spectral gap in the general case

$$S_A(t) = S_{A_0}(t) + \int_0^t S_{A_0}(t-\tau) B S_A(\tau)\,d\tau$$
$$= S_B(t) + \int_0^t S_B(t-\tau) A_0 S_A(\tau)\,d\tau.$$

- Since $\|S_{A_0}(t)\|_{L^p \to L^p} \le \mathrm{e}^{\lambda_1 t}$ and $\lambda_1 < 0$, using the first equality, if we show that the operator R_t defined by

$$R_t u := \int_0^t S_{A_0}(t-\tau) B S_A(\tau) u\,d\tau = \sigma^2 \int_0^t S_{A_0}(t-\tau) J * S_A(\tau) u\,d\tau$$

  is compact from $L^p \longrightarrow L^p$, we may deduce that $\omega_{\mathrm{ess}}(A) < 0$.

- First we show that for $\tau > 0$ fixed the operator $u \mapsto J * S_A(\tau) u$ is compact. (Use the Kolmogorov-M. Riesz-Fréchet characterization of compact sets in L^p).

Spectral gap in the general case

- Then one shows that for $\varepsilon > 0$ the operator

$$u \mapsto \int_{\varepsilon}^{t} S_{A_0}(t-\tau) J * S_A(\tau) u \, d\tau$$

is compact, and then we pass to the limit $\varepsilon \to 0$.

- Finally we deduce

7 Theorem. *There exist $c_0 > 0$ and $a > 0$ such that $u_0 \in L^p(\mathbb{R}^N)$*

$$\|S_A(t)u_0 - (u_0|\varphi)\varphi\|_{L^p} \leq c_0 \, \mathrm{e}^{-at} \, \|u_0 - (u_0|\varphi)\varphi\|_{L^p} \, .$$

- The same result can be proved in the space $C_0(\mathbb{R}^N)$ and the space of bounded measures $\mathcal{M}(\mathbb{R}^N)$.

Linear nonlocal equation: convergence

- We want to study the convergence of positive solutions of the nonlocal equation

$$
(15)\quad \begin{cases} \partial_t u = \sigma^2(J * u - u) - (W(x) - \overline{W}(u))u & \text{in } (0,\infty)\times\mathbb{R}^N \\ u(0,x) = u_0(x) \geq 0 \quad \text{in } \mathbb{R}^N, & M(u_0) = 1. \end{cases}
$$

where $J \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and W satisfy

$$
J \geq 0, \quad \int_{\mathbb{R}^N} J(x)dx = 1, \quad J(-x) = J(x), \qquad \lim_{|x|\to\infty} W(x) = +\infty,
$$

while $\overline{W}(u)$ is a nonlocal term given by

$$
\overline{W}(u) := \overline{W}(u(t)) := \langle u(t,\cdot), W\rangle := \int_{\mathbb{R}^N} W(y)u(t,y)dy.
$$

Linear nonlocal equation: convergence

- We show first a convergence result for the linear equation, upon dropping the term $\overline{W}(u)u$ and setting

(16) $Av := -Lv + \lambda_1 v \quad \text{for } v \in D(A) := D(L), \quad v(t) := e^{(\lambda_1+\sigma^2)t}u(t)$

and considering the equation

(17)
$$\begin{cases} \partial_t v = Av = -Lv + \lambda_1 v & \text{in } (0,\infty)\times\mathbb{R}^N \\ v(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

- Here we have $A\varphi = 0$, and for $v \in D(L) \cap \varphi^\perp$ we know that

$$(Lv|v) \geq \lambda_2 \|v\|^2 ,$$

and thus

$$(Av|v) \leq (\lambda_1 - \lambda_2) \|v\|^2 .$$

Linear nonlocal equation: convergence

- Since $\lambda_1 - \lambda_2 < 0$, we conclude easily that for any $u_0 \in \varphi^\perp$ the solution $v(t)$ of **(17)** converges to zero, and more precisely we have

$$\left\| e^{tA} u_0 \right\| = \| S_A(t) u_0 \| = \| v(t) \| \leq e^{(\lambda_1 - \lambda_2)t} \| u_0 \| .$$

- Consequently for any $u_0 \in L^2(\mathbb{R}^N)$ we have

$$\| S_A(t) u_0 - (u_0 | \varphi) \varphi \| \leq e^{(\lambda_1 - \lambda_2)t} \| u_0 - (u_0 | \varphi) \varphi \| .$$

- The same result can be established in the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$, as well as in $C_0(\mathbb{R}^N)$ and in the space of bounded measures $\mathcal{M}(\mathbb{R}^N)$. ■

Nonlinear nonlocal equation

- Now we turn to the study of the convergence of positive solutions of the nonlinear nonlocal equation

$$(18)\quad \begin{cases} \partial_t u = \sigma^2 (J * u - u) - (W(x) - \overline{W}(u))u & \text{in } (0,\infty)\times\mathbb{R}^N \\ u(0,x) = u_0(x) \geq 0 \quad \text{in } \mathbb{R}^N, & M(u_0) = 1. \end{cases}$$

  where

$$\overline{W}(u) := \overline{W}(u(t)) := \langle u(t,\cdot), W\rangle := \int_{\mathbb{R}^N} W(y) u(t,y) dy.$$

- Recall that we have set

  $$(19)\quad Av := -Lv + \lambda_1 v \quad \text{for } v \in D(A) := D(L), \quad v(t) := \mathrm{e}^{(\lambda_1+\sigma^2)t} u(t)$$

  and that $S_A(t) := \mathrm{e}^{tA}$ is the semigroup generated by A.

Nonlinear nonlocal equation

- One shows the following

8 Lemma. *Assume that*

$$u_0 \geq 0, \qquad (1+W)u_0 \in L^1(\mathbb{R}^N), \qquad M(u_0) = 1.$$

Then the solution of **(18)** *is given by*

$$u(t) = \frac{S_A(t)u_0}{M(S_A(t)u_0)}.$$

Nonlinear nonlocal equation

- We begin by showing that

$$\frac{d}{dt}M(S_A(t)u_0) = M(AS_A(t)u_0) = (\lambda_1 + \sigma^2)M(S_A(t)u_0) - M(WS_A(t)u_0).$$

- Next, using the above calculation we have

$$\begin{aligned}
\partial_t u &= \frac{d}{dt}\frac{S_A(t)u_0}{M(S_A(t)u_0)} = \frac{AS_A(t)u_0}{M(S_A(t)u_0)} - \frac{M(AS_A(t)u_0)}{M(S_A(t)u_0)}u(t). \\
&= Au(t) - (\lambda_1 + \sigma^2)u(t) + M(Wu(t))u(t) \\
&= -Lu(t) - \sigma^2 u(t) + \overline{W}(u(t))u(t) \\
&= \sigma^2(J * u - u) - Wu + \overline{W}(u)u.
\end{aligned}$$

Nonlinear nonlocal equation

- We can now state

> **9 Theorem.** *Set* $\varphi_1 := \varphi / M(\varphi)$*. Then there exists* $a > 0$ *such that for any* $u_0 \geq 0$ *and* $M(u_0) = 1$*, for a constant* $c(u_0)$ *depending on* u_0*, the solution of* **(18)** *satisfies*
>
> $$\|u(t) - \varphi_1\|_{L^1} \leq c(u_0)\mathrm{e}^{-at}.$$

- The result can also be proved for the norms in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$, as well as in $C_0(\mathbb{R}^N)$ and $\mathscr{M}(\mathbb{R}^N)$.

Nonlinear nonlocal equation

- It is sufficient to write $z(t) := u(t) - \varphi_1$ as

$$
\begin{aligned}
z(t) &= \frac{M(\varphi)S_A(t)u_0 - M(S_A(t)u_0)\varphi}{M(S_A(t)u_0)M(\varphi)} \\
&= \frac{(u_0|\varphi)M(\varphi)S_A(t)u_0 - M(S_A(t)u_0)(u_0|\varphi)\varphi}{M(S_A(t)u_0)M(\varphi)(u_0|\varphi)} \\
&= \frac{M\left((u_0|\varphi)\varphi - S_A(t)u_0\right)S_A(t)u_0 + M(S_A(t)u_0)\left(S_A(t)u_0 - (u_0|\varphi)\varphi\right)}{M(S_A(t)u_0)M(\varphi)(u_0|\varphi)}
\end{aligned}
$$

- This yields

$$
\begin{aligned}
\|z(t)\|_{L^1} \le\ & \frac{\|S_A(t)u_0 - (u_0|\varphi)\varphi\|_{L^1}}{M(\varphi)(u_0|\varphi)} \cdot \frac{\|S_A(t)u_0\|_{L^1}}{M(S_A(t)u_0)} \\
& + \frac{\|S_A(t)u_0 - (u_0|\varphi)\varphi\|_{L^1}}{M(\varphi)(u_0|\varphi)}
\end{aligned}
$$

■