Control and Analysis of PDEs

In honour of Marius Tucsnak

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Remarks on the Convergence to equilibrium in some parabolic evolution equations

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Results concerning non local parabolic equations are obtained in collaboration with Matthieu Alfaro (université de Rouen) and Pierre Gabriel Université de Paris–Saclay (site de Versailles), a work in progress:

Confining Integro-Differential Equations Originating from Evolutionary Biology: Ground States and Long Time Dynamics.
Today’s talk

Introduction
Döblin’s theorem
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Spectral gap in the general case
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  Nonlinear nonlocal equation
Consider a Banach space $X$, a subspace of $L^1(d\mu)$, the space of integrable functions on a measurable set $\Omega$ with respect to a positive measure $d\mu$.

Recall that a linear $C_0$ semigroup on $X$ is a family of bounded operators $S(t) : X \rightarrow X$, for $t \geq 0$, such that

$$S(0) = I, \quad S(t + \tau) = S(t)S(\tau), \quad \forall u \in X, \quad \lim_{t \rightarrow 0} S(t)u = u.$$  

Let $(S(t))_{t \geq 0}$ be a linear $C_0$ semigroup acting on $X$ which preserves positivity, that is

$$f \in X, \quad f \geq 0 \quad \Rightarrow \quad S(t)f \geq 0.$$ 

Our aim is to study the limit of $S(t)f$ as $t \rightarrow +\infty$, for a given $f \geq 0$. 

Introduction
The family under study is a non local diffusion equation of the form

\[
\begin{aligned}
\frac{\partial}{\partial t} f &= \sigma^2 (J * f - f) - (W(x) - \overline{W}(f)) f \\
\quad &\quad \text{in } (0, \infty) \times \mathbb{R}^N \\
\quad &\quad f(0, x) = f_0(x) \geq 0 \quad \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} f_0(x) = 1.
\end{aligned}
\]

where \( J \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) and \( W \geq 0 \) are such that

\[
J \geq 0, \quad \int_{\mathbb{R}^N} J(x) dx = 1, \quad J(-x) = J(x), \quad \lim_{|x| \to \infty} W(x) = +\infty,
\]

while \( \overline{W}(u) \) is a nonlocal term defined by

\[
\overline{W}(f) := \overline{W}(f(t)) := \langle f(t, \cdot), W \rangle := \int_{\mathbb{R}^N} W(y) f(t, y) dy.
\]

This equation is considered in the study of replication-mutation mathematical models in biology.
A classical approach is to study the spectrum $\sigma(L)$ of the generator $L$ of the semigroup, that is the operator $L$ such that $S(t) = \exp(tL)$.

For instance, one tries to show that $0 \in \sigma(L)$ and that there exists a non trivial $\varphi \geq 0$ verifying $L\varphi = 0$.

Then one tries to prove that $L$ has a spectral gap: this means that there exists $\omega_0 > 0$ verifying

$$\sigma(L) \setminus \{0\} \subset \{z \in \mathbb{C} ; \text{Re}(z) < -\omega_0\}.$$ 

Finally one shows that $S(t)f_0$ converges to a multiple of $\varphi$ provided $f_0 \geq 0$, with a decay rate of order $e^{-at}$ for some $a$ verifying $0 < a \leq \omega_0$. ■
A very elegant result is due to Wolfgang Döblin (1915 — 1940), sent to the French « Académie des Sciences » in February 1940 in a sealed document (number 11.688), which was unsealed and made public only in 2000.

Consider a positive semigroup \((S(t))_{t \geq 0}\) acting on the space \(L^1(d\mu)\) and such that for \(f \in L^1(d\mu)\) we have

\[
M(S(t)f) = M(f),
\]

where we denote by \(M(f)\) the total mass of \(f\), that is

\[
M(f) := \int f d\mu.
\]

Then we have
Döblin’s Theorem. Assume that there exists $T > 0$, a number $\theta \in (0, 1)$, and a function (or measure...) $\varphi \geq 0$ for which $M(\varphi) = 1$ and such that for all $g \in L^1(d\mu)$, verifying $g \geq 0$, one has

\begin{equation}
S(T)g \geq \theta M(g) \varphi.
\end{equation}

Then for any $f \in L^1(d\mu)$ such that $M(f) = 0$ one has

\begin{equation}
\|S(T)f\|_{L^1(d\mu)} \leq (1 - \theta) \|f\|_{L^1(d\mu)},
\end{equation}

and also

\begin{equation}
\|S(t)f\|_{L^1(d\mu)} \leq e^{-\omega t} \|f\|_{L^1(d\mu)},
\end{equation}

with

$$\omega := \frac{-\log(1 - \theta)}{T} > 0.$$
Döblin’s theorem

Indeed, if $M(f) = 0$, write $f = f^+ - f^-$ with $f^\pm := \max(0, \pm f)$, and set

$$\lambda := M(f^+) = \int f^+ \, d\mu = \int f^- \, d\mu.$$ 

Then

$$\|S(T)f\|_{L^1(d\mu)} = \|(S(T)f^+ - \theta \lambda \varphi) + (\theta \lambda \varphi - S(T)f^-)\|_{L^1(d\mu)}$$

(5)

$$\leq \|S(T)f^+ - \theta \lambda \varphi\|_{L^1(d\mu)} + \|\theta \lambda \varphi - S(T)f^-\|_{L^1(d\mu)}.$$

But thanks to (2) we have $S(T)f^\pm - \theta \lambda \varphi \geq 0$, and so

$$\|S(T)f^\pm - \theta \lambda \varphi\|_{L^1(d\mu)} = \int (S(T)f^\pm - \theta \lambda \varphi) \, d\mu = (1 - \theta)\lambda,$$

which, plugged into (5) and noting that $2\lambda = \|f\|_{L^1(d\mu)}$, yields

$$\|S(T)f\|_{L^1(d\mu)} \leq 2(1 - \theta)\lambda = (1 - \theta) \|f\|_{L^1(d\mu)}.$$
Now the proof of the exponential decay of $S(t)f$ when $M(f) = 0$, is straightforward: indeed since $|S(t)f| \leq S(t)|f|$, we have
\[ \|S(t)f\|_{L^1(d\mu)} \leq \|S(t)|f|\|_{L^1(d\mu)} = \|f\|_{L^1(d\mu)}, \]
and so $S(t)$ is a contractive semigroup in $L^1(d\mu)$.

If $t > 0$, write $t = nT + \tau$ for $\tau \in [0, T)$ and $n \geq 0$ integer. Noting that
\[ c_1 := \sup_{0 \leq \sigma \leq T} \|S(\sigma)\| = 1, \]
and using the semigroup property of $S(t)$ we get
\[ \|S(t)f\|_{L^1(d\mu)} \leq c_1 \|S(T)^n f\|_{L^1(d\mu)} \leq (1 - \theta)^n \|f\|_{L^1(d\mu)}. \]
Finally, since $n = (t - \tau)/T$ we deduce (4).
In particular, if we know that
\[ \forall t \geq 0, \quad S(t)\varphi = \varphi, \quad M(\varphi) = 1, \quad \varphi \geq 0, \]
then applying the above result to the function
\[ f - M(f)\varphi, \]
we conclude that for any initial data \( f \geq 0 \) we have
\[ \|S(t)f - M(f)\varphi\|_{L^1(d\mu)} \leq e^{-\omega t} \|f - M(f)\varphi\|_{L^1(d\mu)}. \]

Despite this result being very elegant and powerful, the main issue is to prove (2), even if one succeeds to prove the existence of the ground state \( \varphi \). Indeed, in most interesting cases, one has \( \varphi > 0 \) in \( \mathbb{R}^N \) and it is not easy to show a uniform lower bound for \( S(t)f \) when \( f \geq 0 \) and \( f \not\equiv 0 \).
We study existence of a positive stationary solution (or ground state) for the equation
\[
\begin{aligned}
\partial_t f &= \sigma^2 (J \ast f - f) - (W(x) - \overline{W}(f)) f \quad \text{in } (0, \infty) \times \mathbb{R}^N \\
f(0, x) &= f_0(x) \geq 0 \quad \text{in } \mathbb{R}^N, \quad M(f_0) = 1.
\end{aligned}
\]

This amounts to find \(f_0 \geq 0\) such that \(f_0 \not\equiv 0\) and
\[
\sigma^2 (J \ast f_0 - f_0) - W(x) f_0 + \overline{W}(f_0) f_0 = 0,
\]
so that setting \(\lambda := \overline{W}(f_0) > 0\), we end up looking for \(f_0 \geq 0\) solution to
\[
-\sigma^2 J \ast f_0 + W f_0 = (\lambda - \sigma^2) f_0.
\]
Nonlocal equation: ground state

This is an eigenvalue problem: if \( \varphi \geq 0 \) is an eigenfunction and \( \lambda_1 \in \mathbb{R} \) are such that

\[
-\sigma^2 J \ast \varphi + W \varphi = \lambda_1 \varphi, \quad \int_{\mathbb{R}^N} |\varphi(x)|^2 \, dx = 1,
\]

then setting

\[
\lambda := \lambda_1 + \sigma^2, \quad f_0 := \alpha \varphi, \quad \text{with} \quad \alpha := \left( \int_{\mathbb{R}^N} \varphi(x) W(x) \, dx \right)^{-1} \lambda,
\]

we have a stationary solution for (6), and \( M(f_0) = 1 \) (actually one has also \( \lambda = \lambda_1 + \sigma^2 > 0 \) so that \( f_0 \geq 0 \)).

Hence we are going to prove that the linear operator associated to equation (6) has a positive eigenfunction, associated to an eigenvalue \( \lambda_1 > -\sigma^2 \).
Assume that $J \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ satisfies

\begin{equation}
J \geq 0, \quad J(x) = J(-x), \quad \int_{\mathbb{R}^N} J(x) \, dx = 1,
\end{equation}

and

\begin{equation}
\exists J_0 > 0, \exists r_0 > 0, \quad J(x) \geq J_0 \quad \text{on} \quad B(0, r_0).
\end{equation}

The potential $W$ satisfies

\begin{equation}
W \in C(\mathbb{R}^N), \quad W \geq 0, \quad \lim_{|x| \to \infty} W(x) = +\infty,
\end{equation}

and

\begin{equation}
\exists x_* \in \mathbb{R}^N, \exists r_0 > 0, \quad W(x_*) = 0, \quad \frac{1}{W} 1_{[W > 0]} \notin L^1(B(x_*, r_0)).
\end{equation}
Then consider the operator \((L, D(L))\) acting on \(L^2(\mathbb{R}^N)\) defined by
\[
Lu := -\sigma^2 J \ast u + Wu, \quad D(L) := L^2(1 + W).
\]
equation (6) can be written as
\[
\partial_t f = -Lf - \sigma^2 f + W(f)f, \quad f(0) = f_0 \geq 0, \quad M(f_0) = 1,
\]
and the eigenvalue problem (7) reads
\[
L\varphi = \lambda_1 \varphi, \quad \|\varphi\|^2 = 1,
\]
and one would like to have \(\varphi \geq 0\) and \(\lambda_1 + \sigma^2 > 0\).
Equation (1) has been widely studied, when instead of the linear non local diffusion $\sigma^2 (J \ast f - f)$ one has a diffusion given for instance by $-(-\Delta)^s$ for $0 < s \leq 1$, or in some special cases of $J$ and $W$ (for instance M. Alfaro, J. Coville, R. Bürger, F. Li, X. Wang, G. Legendre).

For a diffusion equation like

$$\partial_t f = \Delta f - (|x|^4 - |x|^2)f$$

one can show easily that there exists a ground state $\varphi > 0$ and $\lambda_1 \in \mathbb{R}$ such that

$$-\Delta \varphi + (|x|^4 - |x|^2) \varphi = \lambda_1 \varphi.$$ 

More generally the existence of a ground state for equations such as

$$-\Delta \varphi + W \varphi = \lambda_1 \varphi$$

has been widely studied (see M. Reed & B. Simon, L.A. Takhtajan).
2 Theorem. There exists \( \varphi \in C_0(\mathbb{R}^N) \cap L^2(1 + W) \) and \( \lambda_1 \in \mathbb{R} \), an eigenvalue of multiplicity one, such that

\[
-\sigma^2 J \ast \varphi + W \varphi = \lambda_1 \varphi,
\]

and moreover

\[
-\sigma^2 < \lambda_1 < 0, \quad \forall x \in \mathbb{R}^N, \quad 0 < \varphi(x) \leq \frac{\sigma^2 \|J\|_{L^2(\mathbb{R}^N)}}{W(x) - \lambda_1}.
\]
Define

\[ E(u) := (Lu|u) = -\sigma^2 \int_{\mathbb{R}^N} (J * u)(x)u(x)dx + \int_{\mathbb{R}^N} W(x)u^2(x)dx. \]

and the manifold

\[ S := \left\{ u \in L^2(1 + W) : \int_{\mathbb{R}^N} u^2(x)dx = 1 \right\}. \]

We want to show that

\[ \lambda_1 := \inf_{u \in S} E(u) \]

is achieved for some \( \varphi \in S \) such that \( \varphi \geq 0 \).

The difficulty here is that we do not have any compactness in the imbedding \( L^2(1 + W) \subset L^2(\mathbb{R}^N) \).
One shows first that there exists \( u \in S \) such that \( E(u) < 0 \) and thus \( \lambda_1 < 0 \). (This is the crucial step...).

Then one verifies that if \( u_n \rightharpoonup u \) in \( L^2(1 + W) \) one has

\[
\sigma^2 \int_{\mathbb{R}^N} (J * u_n)u_n dx \rightarrow \sigma^2 \int_{\mathbb{R}^N} (J * u)u dx.
\]

This is due to the fact that on the one hand for \( x \in \mathbb{R}^N \) we have \( K(x - \cdot) \in L^2(\mathbb{R}^N) \) and thus \( K * u_n(x) \rightarrow K * u(x) \), and on the other hand \( u_n \in L^2(1 + W) \).

Next observe that \( E(|u|) \leq E(u) \), so that if

\[ \varphi_n \in S \quad \text{satisfies} \quad \lambda_1 \leq E(\varphi_n) \leq \lambda_1 + \frac{1}{n}, \]

we can assume that \( \varphi_n \geq 0 \).
Moreover since
\[
\int W(x)\varphi_n(x)^2\,dx = E(\varphi_n) + \sigma^2 \int (J * \varphi_n)\varphi_n\,dx \leq \lambda_1 + 1 + \sigma^2,
\]
assuming that \( \varphi_n \rightharpoonup \varphi \) in \( L^2(1+W) \)
0 \leq \int W(x)\varphi_n(x)^2\,dx = E(\varphi_n) + \sigma^2 \int (J * \varphi_n)\varphi_n \to \lambda_1 + \sigma^2 \int (J * \varphi)\varphi\,dx,
and by Fatou's lemma
\[
\int W(x)\varphi(x)^2\,dx \leq \lambda_1 + \sigma^2 \int (J * \varphi)\varphi\,dx \leq \lambda_1 + \sigma^2.
\]
Since \( \lambda_1 < 0 \), this implies that \( \varphi \not\equiv 0 \), and hence \( \lambda_1 + \sigma^2 > 0 \).
Now the same inequality shows that

\[ E(\varphi) \leq \lambda_1 \quad \text{and obviously} \quad \int_{\mathbb{R}^N} \varphi(x)^2 \, dx \leq 1. \]

Therefore, if \( \theta := \|\varphi\|_{L^2} < 1 \), upon considering \( \tilde{\varphi} := \varphi / \theta \), we would get

\[ \tilde{\varphi} \in S, \quad \lambda_1 \leq E(\tilde{\varphi}) = \theta^{-2} E(\varphi) \leq \theta^{-2} \lambda_1 < \lambda_1, \]

since \( \lambda_1 < 0 \). This means that we must have \( \|\varphi\|_{L^2} = 1 \) and \( \varphi \in S \) while \( E(\varphi) = \lambda_1 \).

Finally since

(12) \[ \varphi = \frac{\sigma^2 J * \varphi}{W - \lambda_1} \]

and \( J * \varphi \in C_0(\mathbb{R}^N) \), we infer that \( \varphi \in C_0(\mathbb{R}^N) \). \( \square \)
In order to prove the positivity of \( \varphi \), one may use (12), but in order to show the simplicity of \( \lambda_1 \) we need a strong maximum principle for the operator \( L \).

3 **Lemma.** Let \( 1 \leq p \leq \infty \) and \( f \in L^p(\mathbb{R}^N) \). If \( f \geq 0 \) and \( f \not\equiv 0 \) is given and \( u \in L^p(\mathbb{R}^N) \) solves

\[
Lu + \lambda u = f,
\]

for some \( \lambda > \sigma^2 \), then \( u \geq 0 \), and if \( u \in C_0(\mathbb{R}^N) \) we have \( u > 0 \).

When \( p < \infty \), write

\[
\sigma^2 J * u^- + (W + \lambda)u = f + \sigma^2 J * u^+ \geq 0
\]

and then multiply by \((u^-)^{p-1} 1_{\{u<0\}}\) to get

\[
\int_{\mathbb{R}^N} \left( W(x) + \lambda - \sigma^2 \right) |u^-(x)|^p \, dx \leq 0 \quad \implies \quad u^- \equiv 0.
\]
When $p = \infty$, denote

$$m := \text{ess inf}_{x \in \mathbb{R}^N} u(x).$$

Since $J \geq 0$, we have

$$\sigma^2 \int_{\mathbb{R}^N} J(x - y)u(y)dy \geq m\sigma^2 \int J(x - y)dy = m\sigma^2,$$

so that

$$(W(x) + \lambda)u = f + \sigma^2 J * u \geq \sigma^2 m.$$

If $m < 0$, consider a sequence $(x_n)_n$ such that $u(x_n) \to m$ with $m \leq u(x_{n+1}) \leq u(x_n) < 0$, and

$$0 \leq u(x_n) - m \leq \frac{-1}{2\lambda}(\lambda - \sigma^2)m.$$
One would get

\[ 0 \leq -W(x_n)u(x_n) \leq \lambda(u(x_n) - m) + (\lambda - \sigma^2)m \leq \frac{1}{2}(\lambda - \sigma^2)m < 0. \]

Consequently one must have \( m \geq 0 \), that is \( u \geq 0 \) a.e. \( \square \)
In order to prove $\varphi > 0$ note that if $\lambda > \sigma^2$ and $f := (\lambda_1 + \lambda)\varphi$, we have $f \geq 0$ and $f \neq 0$

\[ L\varphi + \lambda\varphi = f \geq 0 \]

therefore $\varphi > 0$, thanks to the previous lemma.

Next one shows:

\[ 4 \text{ Lemma. } \text{The eigenvalue } \lambda_1 \text{ is simple.} \]

\[ \text{Proof.} \]

Otherwise there would exist $v \neq 0$ such that $v^\pm \neq 0$ and $Lv = \lambda_1 v$.

In particular $\lambda_1 \leq E(|v|) \leq E(v) = \lambda_1$, and thus $L |v| = \lambda_1 |v|$. 
Therefore $L\nu^\pm = \lambda_1 \nu^\pm$ and for $\lambda > \sigma^2$ by Lemma 3 we have

$$L\nu^\pm + \lambda \nu^\pm = (\lambda_1 + \lambda) \nu^\pm \geq 0 \implies \nu^\pm > 0 \text{ in } \mathbb{R}^N.$$  

Since $(\nu^+ | \nu^-) = 0$, this is a contradiction.
The eigenfunction $\varphi$ being found with a simple eigenvalue $\lambda_1 < 0$, we define

$$(13) \quad S_0 := \left\{ u \in L^2(1 + W) ; (u|\varphi) = 0 \text{ and } \int_{\mathbb{R}^N} |u(x)|^2 \, dx = 1 \right\},$$

and

$$(14) \quad \lambda_2 := \inf_{u \in S_0} E(u).$$

Then we have

5 Lemma. We have either $\lambda_2 \geq 0$ or there exists $\psi \in S_0$ such that $\lambda_1 < \lambda_2 = E(\psi) < 0$. 

Indeed, if $\lambda_2 < 0$, following the same arguments as in the proof of Theorem 2, one shows that $\lambda_2$ is achieved for some $\psi \in S_0$.

Now, since $\psi \not\equiv 0$, and $(\varphi|\psi) = 0$, we have $\psi^\pm \not\equiv 0$, and
\[
\lambda_2 = E(\psi) = E(\psi^+) + E(\psi^-) + 2\sigma^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(y) \psi^+(x - y) \psi^-(x) dx dy.
\]

However $E(\psi^\pm) > \lambda_1 \|\psi^\pm\|^2$, since $\psi^\pm$ cannot be a multiple of $\varphi$, and
\[
E(\psi^+) + E(\psi^-) > \lambda_1 \|\psi^+\|^2 + \lambda_1 \|\psi^-\|^2 = \lambda_1 \|\psi\|^2 = \lambda_1.
\]

Also clearly
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(y) \psi^+(x - y) \psi^-(x) dx dy \geq 0,
\]
and so $\lambda_2 > \lambda_1$. ■
Spectral gap in the general case

Since \((Av|v) \leq 0\), the semigroup \(S_A(t) := e^{tA}\) is contractive in \(L^2(\mathbb{R}^N)\)
\[
\|S_A(t)\|_{L^2 \rightarrow L^2} \leq 1.
\]

In what follows let \(1 \leq p < \infty\). Recall (cf. the book *One parameter semi-groups* by K.J. Engel & R. Nagel, chapter 4, § 2) that the growth bound of \(S_A\) is defined as being
\[
\omega_0(A) := \inf \left\{ \omega \in \mathbb{R} ; \exists M > 0, \forall t \geq 0, \|S_A(t)\|_{L^p \rightarrow L^p} \leq Me^{\omega t} \right\},
\]
and the spectral bound of \(A\) (which is self-adjoint in \(L^2(\mathbb{R}^N)\)) is
\[
s(A) := \sup \left\{ \text{Re}(\lambda) ; \lambda \in \sigma(A) \right\} = \sup \left\{ \lambda ; \lambda \in \sigma(A) \right\}.
\]

Thus in our case we have \(\omega_0(A) = s(A) = 0\).
We wish to show that there exists \( a > 0 \) such that \( \sigma(A) \setminus \{0\} \subset (−\infty, −a] \), or

\[
\omega_0(A|\varphi^\perp) = s(A|\varphi^\perp) \leq -a.
\]

To this end we shall use the notion of \textit{essential growth bound}, defined by

\[
\omega_{\text{ess}}(A) := \inf \left\{ \omega \in \mathbb{R} ; \exists M > 0, \forall t \geq 0, \|S_A(t)\|_{\text{ess}} \leq Me^{\omega t} \right\},
\]

where the \textit{essential norm} of a bounded operator \( B \) is defined by

\[
\|B\|_{\text{ess}} := \inf \left\{ \|B - T\|_{L^p \rightarrow L^p} ; T : L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \text{ is compact} \right\}.
\]

Clearly \( \omega_{\text{ess}}(A) \leq \omega_0(A) \). One says that \( S_A \) is \textbf{quasi-compact} if \( \omega_{\text{ess}}(A) < 0 \), and the main interest of this notion is the following result (Corollary 2.11, chapitre 4, § 2 of the above book by Engel & Nagel)
6 Theorem. If $\omega > \omega_{\text{ess}}(A)$ then

$$\sigma(A) \cap \{ \lambda \in \mathbb{C} ; \text{Re}(\lambda) \geq \omega \}.$$

is at most a finite number of eigenvalues of $A$, each having a finite multiplicity.

- In our case, $A$ is self-adjoint (in $L^2(\mathbb{R}^N)$) and we show that $\omega_{\text{ess}}(A) < 0$. Then we fix $0 < a < -\omega_{\text{ess}}(A)$ in order to obtain a decay rate of order $e^{-at}$ for $S_A(t)$ in the space $\varphi^\perp$.

- To show that $\omega_{\text{ess}}(A) < 0$, we split $A$ as the sum of two operators $A_0$ and $B$:

$$A = A_0 + B, \quad A_0 u := \lambda_1 u - W u, \quad D(A_0) := D(L), \quad Bu := \sigma^2 J \ast u.$$
We have $S_{A_0}(t)u_0 = e^{t(\lambda_1 - W)}u_0$ and
$$\|S_{A_0}(t)\|_{L^p \to L^p} \leq e^{\lambda_1 t}, \quad \|B\|_{L^p \to L^p} \leq \sigma^2.$$ 

Writing equation (17) in two forms
$$\frac{dv}{dt} = A_0 v + Bv \quad \text{and} \quad \frac{dv}{dt} = Bv + A_0 v,$$
with $v(0) = v_0$, and then considering the mild solution of each of these forms we obtain
$$v(t) = S_{A_0}(t)v_0 + \int_0^t S_{A_0}(t - \tau)Bv(\tau)\,d\tau$$
$$= S_B(t)v_0 + \int_0^t S_B(t - \tau)A_0 v(\tau)\,d\tau,$$
one obtains the Duhamel formula.
\[ S_A(t) = S_{A_0}(t) + \int_0^t S_{A_0}(t - \tau)B S_A(\tau) d\tau \]
\[ = S_B(t) + \int_0^t S_B(t - \tau)A_0 S_A(\tau) d\tau. \]

Since \( \|S_{A_0}(t)\|_{L^p \to L^p} \leq e^{\lambda_1 t} \) and \( \lambda_1 < 0 \), using the first equality, if we show that the operator \( R_t \) defined by

\[ R_t u := \int_0^t S_{A_0}(t - \tau)B S_A(\tau) u d\tau = \sigma^2 \int_0^t S_{A_0}(t - \tau) J * S_A(\tau) u d\tau \]

is compact from \( L^p \to L^p \), we may deduce that \( \omega_{\text{ess}}(A) < 0 \).

First we show that for \( \tau > 0 \) fixed the operator \( u \mapsto J * S_A(\tau) u \) is compact. (Use the Kolmogorov-M. Riesz-Fréchet characterization of compact sets in \( L^p \)).
Then one shows that for $\varepsilon > 0$ the operator

$$u \mapsto \int_{\varepsilon}^{t} S_{A_0}(t - \tau) J * S_A(\tau) u d\tau$$

is compact, and then we pass to the limit $\varepsilon \to 0$.

Finally we deduce

**Theorem.** There exist $c_0 > 0$ and $a > 0$ such that $u_0 \in L^p(\mathbb{R}^N)$

$$\|S_A(t)u_0 - (u_0|\varphi)\varphi\|_{L^p} \leq c_0 e^{-at} \|u_0 - (u_0|\varphi)\varphi\|_{L^p}.$$

The same result can be proved in the space $C_0(\mathbb{R}^N)$ and the space of bounded measures $\mathcal{M}(\mathbb{R}^N)$. □
We want to study the convergence of positive solutions of the nonlocal equation

\[
\begin{align*}
\partial_t u &= \sigma^2 (J \ast u - u) - (W(x) - W(u))u \quad \text{in } (0, \infty) \times \mathbb{R}^N \\
u(0, x) &= u_0(x) \geq 0 \quad \text{in } \mathbb{R}^N, \quad M(u_0) = 1.
\end{align*}
\]

where $J \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $W$ satisfy

\[
J \geq 0, \quad \int_{\mathbb{R}^N} J(x)dx = 1, \quad J(-x) = J(x), \quad \lim_{|x| \to \infty} W(x) = +\infty,
\]

while $W(u)$ is a nonlocal term given by

\[
W(u) := W(u(t)) := \langle u(t, \cdot), W \rangle := \int_{\mathbb{R}^N} W(y)u(t, y)dy.
\]
We show first a convergence result for the linear equation, upon dropping the term $W(u)u$ and setting

\[
A_v := -L_v + \lambda_1 v \quad \text{for } v \in D(A) := D(L), \quad v(t) := e^{(\lambda_1 + \sigma^2)t} u(t)
\]

and considering the equation

\[
\begin{aligned}
\partial_t v &= Av = -L_v + \lambda_1 v \quad \text{in } (0, \infty) \times \mathbb{R}^N \\
v(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]

Here we have $A\varphi = 0$, and for $v \in D(L) \cap \varphi^\perp$ we know that

\[
(Lv|v) \geq \lambda_2 \|v\|^2,
\]

and thus

\[
(Av|v) \leq (\lambda_1 - \lambda_2) \|v\|^2.
\]
Since $\lambda_1 - \lambda_2 < 0$, we conclude easily that for any $u_0 \in \varphi^\perp$ the solution $\nu(t)$ of (17) converges to zero, and more precisely we have
\[
\|e^{tA}u_0\| = \|S_A(t)u_0\| = \|\nu(t)\| \leq e^{(\lambda_1 - \lambda_2)t} \|u_0\|.
\]
Consequently for any $u_0 \in L^2(\mathbb{R}^N)$ we have
\[
\|S_A(t)u_0 - (u_0|\varphi)\varphi\| \leq e^{(\lambda_1 - \lambda_2)t} \|u_0 - (u_0|\varphi)\varphi\|.
\]
The same result can be established in the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$, as well as in $C_0(\mathbb{R}^N)$ and in the space of bounded measures $\mathcal{M}(\mathbb{R}^N)$.
Now we turn to the study of the convergence of positive solutions of the nonlinear nonlocal equation

\[
\begin{aligned}
\begin{cases}
\partial_t u = \sigma^2 (J \ast u - u) - (W(x) - \overline{W}(u))u & \text{in } (0, \infty) \times \mathbb{R}^N \\
u(0, x) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N, \quad M(u_0) = 1.
\end{cases}
\end{aligned}
\]

where

\[
\overline{W}(u) := \overline{W}(u(t)) := \langle u(t, \cdot), W \rangle := \int_{\mathbb{R}^N} W(y) u(t, y) dy.
\]

Recall that we have set

\[
\begin{aligned}
A v := -Lv + \lambda_1 v & \text{ for } v \in D(A) := D(L), \quad v(t) := e^{(\lambda_1 + \sigma^2)t} u(t) \\
\text{and that } S_A(t) := e^{tA} \text{ is the semigroup generated by } A.
\end{aligned}
\]
One shows the following

8 Lemma. Assume that

\[ u_0 \geq 0, \quad (1 + W)u_0 \in L^1(\mathbb{R}^N), \quad M(u_0) = 1. \]

Then the solution of (18) is given by

\[ u(t) = \frac{S_A(t)u_0}{M(S_A(t)u_0)}. \]
We begin by showing that

\[
\frac{d}{dt} M(S_A(t)u_0) = M(AS_A(t)u_0) = (\lambda_1 + \sigma^2)M(S_A(t)u_0) - M(WS_A(t)u_0).
\]

Next, using the above calculation we have

\[
\partial_t u = \frac{d}{dt} \frac{S_A(t)u_0}{M(S_A(t)u_0)} = \frac{AS_A(t)u_0}{M(S_A(t)u_0)} - \frac{M(AS_A(t)u_0)}{M(S_A(t)u_0)} u(t).
\]

\[
= Au(t) - (\lambda_1 + \sigma^2)u(t) + M(Wu(t))u(t)
\]

\[
= -Lu(t) - \sigma^2 u(t) + \overline{W}(u(t))u(t)
\]

\[
= \sigma^2 (J \ast u - u) - Wu + \overline{W}(u)u.
\]
We can now state

9 Theorem. Set $\varphi_1 := \varphi / M(\varphi)$. Then there exists $a > 0$ such that for any $u_0 \geq 0$ and $M(u_0) = 1$, for a constant $c(u_0)$ depending on $u_0$, the solution of (18) satisfies

$$\|u(t) - \varphi_1\|_{L^1} \leq c(u_0)e^{-at}. $$

The result can also be proved for the norms in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$, as well as in $C_0(\mathbb{R}^N)$ and $M(\mathbb{R}^N)$. 
Nonlinear nonlocal equation

- It is sufficient to write $z(t) := u(t) - \varphi_1$ as

$$z(t) = \frac{M(\varphi)S_A(t)u_0 - M(S_A(t)u_0)\varphi}{M(S_A(t)u_0)M(\varphi)}$$

$$= \frac{(u_0|\varphi)M(\varphi)S_A(t)u_0 - M(S_A(t)u_0)(u_0|\varphi)\varphi}{M(S_A(t)u_0)M(\varphi)(u_0|\varphi)}$$

$$= \frac{M((u_0|\varphi)\varphi - S_A(t)u_0)S_A(t)u_0 + M(S_A(t)u_0)(S_A(t)u_0 - (u_0|\varphi)\varphi)}{M(S_A(t)u_0)M(\varphi)(u_0|\varphi)}$$

- This yields

$$\|z(t)\|_{L^1} \leq \frac{\|S_A(t)u_0 - (u_0|\varphi)\varphi\|_{L^1}}{M(\varphi)(u_0|\varphi)} \cdot \frac{\|S_A(t)u_0\|_{L^1}}{M(S_A(t)u_0)}$$

$$+ \frac{\|S_A(t)u_0 - (u_0|\varphi)\varphi\|_{L^1}}{M(\varphi)(u_0|\varphi)}$$