Controllability and Riesz bases of infinite-dimensional port-Hamiltonian systems

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dedicated to Marius Tucsnak on the occation of his 60th birthday

MATHEMATICAL MODELLING, ANALYSIS AND COMPUTATIONAL MATHEMATICS

BERGISCHE UNIVERSITÄT WUPPERTAL

Linear finite-dimensional port-Hamiltonian systems (PHS)

$$\dot{x}(t) = (J - R)\mathcal{H}x(t) + Gu(t)$$
$$y(t) = G^T\mathcal{H}x(t)$$

Hamiltonian: $H(x) = \frac{1}{2}x^T\mathcal{H}x$ State space: $\mathbb{R}^n$,

$\mathcal{H} \in \mathbb{R}^{n\times n}$ positive definite, $J^T = -J$ and $R \geq 0$.

For every $x(0) \in \mathbb{R}^n$ there exists a unique solution and

$$\dot{H}(x(t)) = -(\mathcal{H}x(t))^T R\mathcal{H}x(t) + (y(t))^T u(t) \leq (y(t))^T u(t).$$

(Maschke and van der Schaft '92)

A general linear infinite-dimensional PHS can be written in a similar form, where the state space is an arbitrary Hilbert space and J, R, $\mathcal{H}$ and G are operators.

Example: Wave equation

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

$$H(w(\cdot,t)) = \frac{1}{2}\int_0^1 \rho(\zeta)\left[\frac{\partial w}{\partial t}(\zeta,t)\right]^2 + T(\zeta)\left[\frac{\partial w}{\partial \zeta}(\zeta,t)\right]^2 d\zeta$$

$x_1 := \rho\frac{\partial w}{\partial t}$ (the momentum), $x_2 := \frac{\partial w}{\partial \zeta}$ (the strain)

$$H\left[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\cdot,t)\right] = \frac{1}{2}\int_0^1 \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} d\zeta$$

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t) = \left(\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P_1}\frac{\partial}{\partial \zeta}\right)\left[\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{\mathcal{H}(\zeta)}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t)\right]$$

Example: The Timoshenko beam

$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial \zeta}\left[K(\zeta)\left[\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right]\right]$$

$$I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial \zeta}\left[EI(\zeta)\frac{\partial \phi}{\partial \zeta}\right] + K(\zeta)\left[\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right]$$

$w(\zeta,t) =$ is transverse displacement of the beam
$\phi(\zeta,t) =$ is rotation angle of a filament of the beam

$$\begin{aligned}
x_1(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t) && \text{shear displacement} \\
x_2(\zeta,t) &= \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) && \text{momentum} \\
x_3(\zeta,t) &= \frac{\partial \phi}{\partial \zeta}(\zeta,t) && \text{angular displacement} \\
x_4(\zeta,t) &= I_\rho(\zeta)\frac{\partial \phi}{\partial t}(\zeta,t) && \text{angular momentum}
\end{aligned}$$

Example: The Timoshenko beam

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= \left[P_1\frac{\partial}{\partial\zeta}+P_0\right][\mathcal{H}(\zeta)x(\zeta,t)]\\ H(x(\cdot,t)) &= \frac{1}{2}\int_0^1 x(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t)d\zeta.\end{aligned}$$

with $P_1 = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$ $P_0 = \begin{bmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}$

$$\mathcal{H}(\zeta) = \begin{bmatrix}K(\zeta)&0&0&0\\0&\frac{1}{\rho(\zeta)}&0&0\\0&0&EI(\zeta)&0\\0&0&0&\frac{1}{I_\rho(\zeta)}\end{bmatrix}$$

Infinite-dimensional PHS

$$\frac{\partial x}{\partial t}(\zeta,t) = \underbrace{\left[P_1\frac{\partial}{\partial\zeta} + P_0\right]}_{J-R}[\mathcal{H}(\zeta)x(\zeta,t)]$$

$$H(x(\cdot,t)) = \frac{1}{2}\int_0^1 x(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t)d\zeta$$

- $P_1 \in \mathbb{R}^{n\times n}$ invertible, symmetric, $P_0 \in \mathbb{R}^{n\times n}$ anti-symmetric,
- $\mathcal{H}(\zeta) \in \mathbb{R}^{n\times n}$ symmetric, invertible with $mI \leq \mathcal{H}(\zeta) \leq MI$ for $m, M > 0$.

$x(\zeta,t) \in \mathbb{R}^n$ and $x(t) := x(\cdot,t) \in L^2([0,1];\mathbb{R}^n)$

State space $X := L^2([0,1];\mathbb{R}^n)$, $\langle x,y\rangle = \frac{1}{2}\int_0^1 x(\zeta)^T\mathcal{H}(\zeta)y(\zeta)\,d\zeta$.

$$\dot{x}(t) = \left[P_1\tfrac{\partial}{\partial\zeta} + P_0\right][\mathcal{H}x(t)]$$

(Le Gorrec, Zwart, Maschke, Villegas, van der Schaft '05)

Infinite-dimensional PHS

Boundary effort e_∂ and boundary flow f_∂:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} [\mathcal{H}x](1) \\ [\mathcal{H}x](0) \end{pmatrix}.$$

$$\frac{\partial x}{\partial t}(\zeta, t) = (P_1 \frac{\partial}{\partial \zeta} + P_0)\,[\mathcal{H}(\zeta)x(\zeta, t)]$$

$$H(x(\cdot, t)) = \frac{1}{2} \int_0^1 x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta$$

$$u(t) = W_B \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}, \quad y(t) = W_C \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

with $W_B, W_C \in \mathbb{R}^{n \times 2n}$ and $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ invertible. Then:

$$\frac{dH}{dt}(x(\cdot, t)) = \frac{1}{2} \left[[\mathcal{H}x]^T (\zeta, t) P_1 [\mathcal{H}x](\zeta, t) \right]_0^1 = e_\partial(t)^T f_\partial(t) \leq u(t)^T y(t)$$

if $\begin{pmatrix} W_B \Sigma W_B^T & W_B \Sigma W_C^T \\ W_C \Sigma W_B^T & W_C \Sigma W_C^T \end{pmatrix}^{-1} \leq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$.

Infinite-dimensional PHS

$$\frac{\partial x}{\partial t}(\zeta,t)=(P_1\frac{\partial}{\partial\zeta}+P_0)[\mathcal{H}(\zeta)x(\zeta,t)],\ u(t)=W_B\begin{bmatrix} f_\partial(t)\\ e_\partial(t)\end{bmatrix},\ y(t)=W_C\begin{bmatrix} f_\partial(t)\\ e_\partial(t)\end{bmatrix}$$

The system is called

- impedance passive, if $\frac{dH}{dt}(x(\cdot,t)) \leq u(t)^T y(t)$.
- impedance energy preserving, if $\frac{dH}{dt}(x(\cdot,t)) = u(t)^T y(t)$.

Theorem

- *The system is impedance passive* $\iff$
$$\begin{bmatrix} W_B\Sigma W_B^T & W_B\Sigma W_C^T\\ W_C\Sigma W_B^T & W_C\Sigma W_C^T\end{bmatrix}^{-1} \leq \begin{bmatrix} 0 & I\\ I & 0\end{bmatrix}.$$
- *The system is impedance energy preserving* $\iff$
$$\begin{bmatrix} W_B\Sigma W_B^T & W_B\Sigma W_C^T\\ W_C\Sigma W_B^T & W_C\Sigma W_C^T\end{bmatrix} = \begin{bmatrix} 0 & I\\ I & 0\end{bmatrix}.$$
- *If the system is impedance passive, then* $W_B\Sigma W_B^T \geq 0$.

In the following we always assume $W_B\Sigma W_B^T \geq 0$.

Infinite-dimensional PHS

$$\frac{\partial x}{\partial t}(\zeta,t)=(P_1\frac{\partial}{\partial\zeta}+P_0)[\mathcal{H}(\zeta)x(\zeta,t)],\ u(t)=W_B\begin{bmatrix}f_\partial(t)\\e_\partial(t)\end{bmatrix},\ y(t)=W_C\begin{bmatrix}f_\partial(t)\\e_\partial(t)\end{bmatrix}$$

$$X = L^2([0,1];\mathbb{R}^n), \qquad \langle x,y\rangle = \tfrac{1}{2}\int_0^1 x(\zeta)^T\mathcal{H}(\zeta)y(\zeta)\,d\zeta$$

$$Ax = \left[P_1\tfrac{d}{d\zeta}+P_0\right][\mathcal{H}x]$$

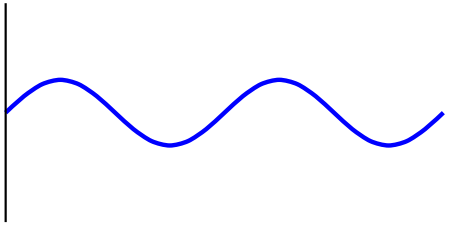
$$D(A) = \left\{x\in X \mid \tfrac{d}{d\zeta}\mathcal{H}x\in X, W_B\begin{bmatrix}f_\partial\\e_\partial\end{bmatrix}=0\right\}$$

Theorem (Le Gorrec, Zwart, Maschke '05)

A *generates a contraction semigroup*

A *generates a unitary group* $\iff W_B\Sigma W_B^T=0$

Example: Wave equation



$$\begin{aligned}\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial\zeta}\left[T(\zeta)\frac{\partial w}{\partial\zeta}(\zeta,t)\right]\\ \frac{\partial w}{\partial t}(0,t) &= 0, \quad T(1)\frac{\partial w}{\partial\zeta}(1,t)=0\end{aligned}$$

$$f_\partial = \frac{1}{\sqrt{2}}\begin{bmatrix} T(1)\frac{\partial w}{\partial\zeta}(1) - T(0)\frac{\partial w}{\partial\zeta}(0) \\ \frac{\partial w}{\partial t}(1) - \frac{\partial w}{\partial t}(0)\end{bmatrix}$$

$$e_\partial = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\partial w}{\partial t}(1) + \frac{\partial w}{\partial t}(0) \\ T(1)\frac{\partial w}{\partial\zeta}(1) + T(0)\frac{\partial w}{\partial\zeta}(0)\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix} T(1)\frac{\partial w}{\partial\zeta}(1,t) \\ \frac{\partial w}{\partial t}(0,t)\end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & -1 & 1 & 0\end{bmatrix}}_{=W_B}\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}$$

rank $W_B = 2$ and $W_B \Sigma {W_B}^T = 0$. Thus A gen. a unitary group.

Well-posedness of PHS

$\frac{\partial x}{\partial t}(\zeta,t)=(P_1\frac{\partial}{\partial\zeta}+P_0)[\mathcal{H}(\zeta)x(\zeta,t)]$, $u(t)=W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}$, $y(t)=W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}$

$P_1\mathcal{H}$ can be factorized as $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$.

Theorem (Zwart, Le Gorrec, Maschke, Villegas '10)

If Δ, S are continuously differentiable, then

- $Ax = (P_1\frac{d}{d\zeta} + P_0)[\mathcal{H}x]$ *with* $D(A) = \{x \in X \mid (\mathcal{H}x)' \in X, W_B\left[\begin{smallmatrix} f_\partial \\ e_\partial\end{smallmatrix}\right] = 0\}$ *generates a contraction semigroup on* X.
- *There are* $t_0, m_{t_0} > 0$*: Every classical solution satisfies* $\|x(t_0)\|_X^2 + \int_0^{t_0}\|y(t)\|^2dt \le m_{t_0}\left[\|x(0)\|_X^2 + \int_0^{t_0}\|u(t)\|^2dt\right]$.
- *For every initial condition* $x_0 \in X$ *and input function* $u \in L^2_{loc}([0,\infty),\mathbb{R}^n)$ *the system has a (mild) solution* $x \in C([0,\infty),X)$ *and* $y \in L^2_{loc}([0,\infty),\mathbb{R}^n)$.

Exact controllability of PHS

$$\frac{\partial x}{\partial t}(\zeta,t)=(P_1\frac{\partial}{\partial\zeta}+P_0)[\mathcal{H}(\zeta)x(\zeta,t)],\ u(t)=W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix},\ y(t)=W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}$$

A system is exactly controllable if

$\exists\,\tau>0\ \forall x_1\in X\ \exists\,u\in L^2(0,\tau,\mathbb{R}^n)$: mild solution x satisfies:

$x(0)=0$ and $x(\tau)=x_1$.

Theorem (J., Zwart, 2018)

Every impedance energy preserving port-Hamiltonian system is exactly controllable.

Sketch of the proof:

- $-A$ generates a contraction semigroup.
- Exact controllability is equivalent to optimizability (Rebarber, Weiss '97).
- System is optimizable by $u(t)=-ky(t)$, $k>0$. (Humaloja, Paunonen '18)

Exact controllability of PHS

$$\frac{\partial x}{\partial t}(\zeta,t)=(P_1\frac{\partial}{\partial\zeta}+P_0)[\mathcal{H}(\zeta)x(\zeta,t)],\ u(t)=W_B\begin{bmatrix} f_\partial(t)\\ e_\partial(t)\end{bmatrix},\ y(t)=W_C\begin{bmatrix} f_\partial(t)\\ e_\partial(t)\end{bmatrix}$$

Main Theorem (K., Jacob 2019)

Every port-Hamiltonian system is exactly controllable.

Sketch of the proof:

Open-loop system exactly controllable $\Longleftrightarrow$ Closed loop system exactly controllable (Weiss '94)

$W_B = \begin{bmatrix} W_1 & W_0 \end{bmatrix}$

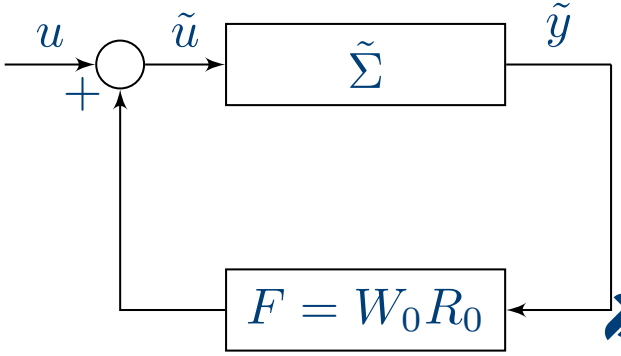
Exist $R_1, R_0 \in \mathbb{R}^{n\times n}$ invertible:

$W_B \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = I.$

$\tilde{\Sigma}$ is a port-Hamiltonian system with

$\tilde{u}(t) = \begin{bmatrix} R_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},$

$\tilde{y}(t) = \begin{bmatrix} R_1^{-1} & R_0^{-1} \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}$



The system operator A

$$Ax = \left(P_1 \tfrac{d}{d\zeta} + P_0\right)[\mathcal{H}x], \; D(A) = \left\{x \in X \mid (\mathcal{H}x)' \in X, W_B \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] = 0\right\}$$

Theorem

- *Spectrum* $\sigma(A) = \{\lambda_k\}$, λ_k *isolated eigenvalue with finite multipl. (Augner '16)*
- $\operatorname{Re}\lambda_k \leq 0$.
- $\{\lambda_k\}$ *can be decomposed in* n *interpolating sequences* ($\inf_k \prod_{i \neq k} \left|\frac{\mu_i - \mu_k}{\mu_i + \overline{\mu_k}}\right|$). *(J., Zwart '01)*
- *If* A *generates a* C_0*-group, then* $-\infty < \inf \operatorname{Re}\lambda_k \leq \sup \operatorname{Re}\lambda_k \leq 0$ *and*

$$X = \overline{\operatorname{span}_{k \in \mathbb{N}} E(\lambda_k) X},$$

  where $E(\lambda_k)$ *are the spectral projections. (J., Zwart '99)*
- *There are port-Hamiltonian systems with* $\sigma(A) = \emptyset$. *E.g.* $Ax = \frac{d}{d\zeta}x$ *with* $D(A) = \{x \in X \mid x' \in X, x(1) = 0\}$

Discrete Riesz spectral operator

$Ax = (P_1 \frac{d}{d\zeta} + P_0)\,[\mathcal{H}x]$ with
$D(A) = \{x \in X \mid (\mathcal{H}x)' \in X, W_B \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] = 0\}$

Theorem

$W_B \Sigma W_B^T = 0 \implies$ Normalized eigenvectors of A form an ONB.

Question: Does a similar result hold for impedance passive system?

In general no: There are port-Hamiltonian systems with $\sigma(A) = \emptyset$.
E.g. $Ax = \frac{d}{d\zeta} x$ with $D(A) = \{x \in X \mid x' \in X, x(1) = 0\}$

Discrete Riesz spectral operator

Let A be an operator with compact resolvent and $\sigma(A) = (\lambda_k)_{k\in\mathbb{N}}$.

$E_k := E((\lambda_k))$, spectral projection regarding the kth eigenvalue.

Definition

A is a *discrete Riesz spectral operator*, if

1. for every $k \in \mathbb{N}$ there exists $N_k \in \mathcal{L}(X)$ such that

$$AE_k = \lambda_k E_k + N_k,$$

2. the sequence of closed subspaces $(E_k(X))_{k\in\mathbb{N}}$ is a *Riesz basis of subspaces of* X, that is, $\operatorname{span}(E_k(X))_{k\in\mathbb{N}}$ is dense and there exists an isomorphism $T \in \mathcal{L}(X)$, such that $(TE_k(X))_{k\in\mathbb{N}}$ is system of pairwise orthogonal subspaces of X.
3. $N := \sum_{k\in\mathbb{N}} N_k$ is bounded and nilpotent.

Discrete Riesz spectral operator

$Ax = (P_1 \frac{d}{d\zeta} + P_0)\,[\mathcal{H}x]$ with
$D(A) = \{x \in X \mid (\mathcal{H}x)' \in X, W_B \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] = 0\}$

$\left[\begin{smallmatrix} W_1 & W_0 \end{smallmatrix}\right] := W_B \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]$

$Z^-(1) :=$ span of the eigenv. of $P_1\mathcal{H}(1)$ to its negative eigenvalues
$Z^+(0) :=$ span of the eigenv. of $P_1\mathcal{H}(0)$ to its positive eigenvalues

Theorem

The following are equivalent:

1. A is a discrete Riesz spectral operator.
2. $-A$ is the generator of a C_0-semigroup.
3. $W_1\mathcal{H}(1)Z^-(1) \oplus W_0\mathcal{H}(0)Z^+(0) = \mathbb{R}^n$.

Sketch of the proof:

2. $\iff$ 3.: J. Morris, Zwart '15

Discrete Riesz spectral operator

Theorem

The following are equivalent:

1. A is a discrete Riesz spectral operator.
2. $-A$ is the generator of a C_0-semigroup.
3. $W_1\mathcal{H}(1)Z^-(1) \oplus W_0\mathcal{H}(0)Z^+(0) = \mathbb{C}^n$.

Sketch of the proof:

2. $\implies$ 1.: $\sigma(A) = (\lambda_k)_k$ counted with algebraic multiplicity.
   - $X = \overline{\operatorname{span}_{k\in\mathbb{N}} E(\lambda_k)X}$, where $E(\lambda_k)$ are the spectral projections. (J., Zwart '99)
   - Exists a sine-type function with zeros $(\lambda_k)_k$
   - $(\lambda_k)_k$ can be decomposed into finitely many set having a uniform gap. (Levin '61)
   - Zwart '10 implies the statement.

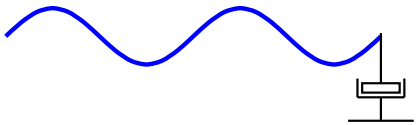
1. $\implies$ 2.: Resolvent estimates and perturbation theory.

Spectrum of discrete Riesz spectral PH operators

Assume A is a discrete Riesz spectral operator.

- $\sigma(A) = \sigma_p(A) = (\lambda_k)_{k\in\mathbb{N}}$ lie in a strip parallel to the imaginary axis.
- the eigenvalues (counted according to the algebraic multiplicity) can be decomposed into finitely many sets each having a uniform gap, i.e., $\inf_{k\neq m} |\mu_k - \mu_m| > 0$.
- A satisfies the spectrum determined growth assumption, that is, $\omega_0(A) = s(A)$. (Guo, Zwart '01)

Example: A damped wave equation



$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=P_1} \frac{\partial}{\partial \zeta}\left[\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{=\mathcal{H}} x(\zeta,t)\right]$$

$$u(t) = \begin{bmatrix} \kappa & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix},$$

$$\kappa > 0, \quad T(0) \neq \kappa\gamma(0).$$

$W_B \Sigma W_B^T \geq 0$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma(1) \\ T(1) \end{bmatrix}$ and $\begin{bmatrix} \kappa & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\gamma(0) \\ T(0) \end{bmatrix}$ are lin. independent.

Thus, the port-Hamiltonian operator A is a discrete Riesz spectral operator.

Remark: Xu, Weiss '11 proved the result for constant functions ρ and T only.

Example: A damped Timoshenko-beam

Timoshenko-beam clamped at $\zeta = 0$ and controlled at $\zeta = 1$ by momentum and angular momentum feedback can be modelled by

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathcal{H}(\zeta) = \begin{bmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 1 \\ 0 & 0 & 0 & \frac{1}{I\rho(\zeta)} \end{bmatrix} \text{ and}$$

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} \text{ with } \begin{bmatrix} W_1 & W_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are given gain feedback constants.

Associated system operator is a discrete Riesz spectral operator.

Remark: If the physical constants are independent of ζ, then Xu, Feng '02 showed that the eigenvectors and generalized eigenvectors form a Riesz basis.

Thanks for your attention!
Happy Birthday Marius!