Control of solids and vortices in a perfect fluid

Olivier Glass

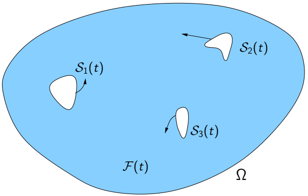
Ceremade, Université Paris-Dauphine-PSL

Control and Analysis of PDE systems, Bordeaux, November 2021

In honor of Marius Tucsnak for his 60th birthday

Part I. Presentation of the main model: rigid bodies immersed in an incompressible perfect fluid

- We consider the motion of rigid bodies immersed in an incompressible perfect fluid in a regular domain $\Omega \subset \mathbb{R}^2$.



Ω is a bounded, regular, connected and simply connected domain.

- The solids occupy at each instant $t \geq 0$ a closed subset $\mathcal{S}_i(t) \subset \Omega$, and the fluid occupies $\mathcal{F}(t) := \Omega \setminus \cup_{i=1}^{N} \mathcal{S}_i(t)$.
  We will not consider collisions:

$$d(\mathcal{S}_i(t), \Omega) > 0 \text{ and for } i \neq j,\ d(\mathcal{S}_i(t), \mathcal{S}_j(t)) > 0$$

Fluid equation

- In $\mathcal{F}(t)$, the fluid satisfies the Euler equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \quad t \in [0, T],\ x \in \mathcal{F}(t), \\ \\ \text{div } u = 0 \qquad t \in [0, T],\ x \in \mathcal{F}(t), \\ \end{cases}$$

  where

  - $u = u(t, x) : x \in \mathcal{F}(t) \to \mathbb{R}^2$ is the fluid velocity,
  - $p = p(t, x) : x \in \mathcal{F}(t) \to \mathbb{R}$ denotes the pressure.

Boundary conditions

- At the boundaries, the fluid satisfies the no-penetration/slip condition :

$$u \cdot n = 0 \text{ for } x \in \partial\Omega,$$
$$u \cdot n = V_{\mathcal{S}_i} \cdot n \text{ for } x \in \partial\mathcal{S}_i(t),$$

where n is the normal to the boundaries $\partial\Omega$ and $\partial\mathcal{S}_i(t)$, and

$$V_{\mathcal{S}_i}(t,x) = h_i'(t) + \vartheta_i'(t)(x - h_i(t))^\perp$$

is the velocity of the i-th body, where:

  - $h_i(t)$ is the position of its center of mass,
  - ϑ_i is the angle with respect to the initial position (so $\vartheta_i(0) = 0$).

Dynamics of the solid

- The dynamics of the i-th solid is driven by the action of the pressure on its surface:

$$m_i \, h_i''(t) = \int_{\partial \mathcal{S}_i(t)} p \, n \; ds,$$

$$\mathcal{J}_i \, \vartheta_i''(t) = \int_{\partial \mathcal{S}_i(t)} p \, (x - h_i(t))^{\perp} \cdot n \; ds,$$

  where
  - $m_i > 0$ is the mass of the i-th body,
  - $\mathcal{J}_i > 0$ denotes its moment of inertia.

- Remark. D'Alembert's paradox does not apply here, because it concerns a single body in a potential fluid in $\mathbb{R}^2$, stationary and constant at infinity. In that case (only), D'Alembert's paradox states that the fluid does not influence the dynamics of the solid.

Other formulations

- Vorticity formulation. In 2-D, the fluid part of the system can also be written

$$\partial_t \omega + (u \cdot \nabla)\omega = 0 \ \text{ in } \ \mathcal{F}(t),$$

and

$$\begin{gathered} \text{curl } u = \omega \ \text{ in } \ \mathcal{F}(t), \\ \text{div } u = 0 \ \text{ in } \ \mathcal{F}(t), \\ \oint_{\partial \mathcal{S}_i(t)} u \cdot \tau \, ds = \oint_{\partial \mathcal{S}_i(0)} u_0 \cdot \tau \, ds = \gamma_i \ \text{ (Kelvin's law)}, \end{gathered}$$

$+$ boundary conditions on $u \cdot n$.

- As for the Euler equation alone, the complete system can be viewed as an equation of geodesics on an infinite dimensional Riemannian manifold, in the spirit of Arnold's work, see also Ebin-Marsden. (G.-Sueur)

References for the Cauchy problem

- Classical solutions (say at least C^1) solutions with finite energy:
  - Ortega-Rosier-Takahashi
  - Rosier-Rosier
  - Houot-San Martin-Tucsnak
- Weaker solutions (Yudovich or DiPerna-Majda type solutions):
  - G.-Sueur
  - Yun Wang-Zhouping Xin

Cauchy problem (2D, Yudovich-type solutions)

Theorem (G.-Sueur)

Let $\mathcal{S}_{i,0} \subset \Omega$ be given. For any $u_0 \in C^0(\overline{\mathcal{F}}_0; \mathbb{R}^2)$, $(h'_{i,0}, \vartheta'_{i,0}) \in \mathbb{R}^{3N}$ such that

$$\textit{div } u_0 = 0, \ \textit{curl } u_0 = \omega_0 \in L^\infty(\mathcal{F}_0),$$
$$u_0 \cdot n = (h'_{i,0} + \vartheta'_{i,0}(x - h_{i,0})^\perp) \cdot n \textit{ on } \partial \mathcal{S}_i(0), \quad u_0 \cdot n = 0 \textit{ on } \partial\Omega.$$

there exists a unique maximal solution of the system:

$$((h_i)_{i=1\dots N}, (\theta_i)_{i=1\dots N}, u) \in C^2([0, T^*); \mathbb{R}^{3N}) \times L^\infty([0, T^*); \mathcal{LL}(\mathcal{F}(t))),$$

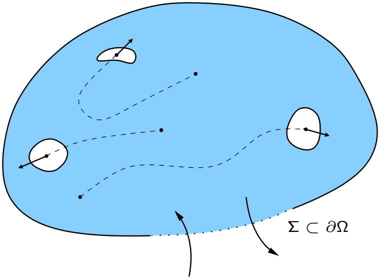
where $T^* \in (0, +\infty]$ is the first possible collision time.

Here $\mathcal{LL}(U) := \Big\{ f \in C^0(U) \ / \ \exists C > 0, \ \forall x, y \in U,$

$$|f(x) - f(y)| \leq C|x-y|(1 + \ln^- |x-y|) \Big\}.$$

Part II. Controlling solids in a perfect fluid

- We consider the same system equipped with a boundary control.
- We consider Σ an open non empty part of the boundary, and aim at understanding how the system can be influenced through non-homogeneous boundary conditions on Σ.



Non-homogeneous boundary conditions for the Euler equation

- Seminal work by Yudovich (1964)
- One can impose the normal part of the velocity field on the boundary, here on Σ. This gives

  $$u \cdot n = g \ \text{ on } \ \Sigma \ \text{ and } \ u \cdot n = 0 \ \text{ on } \ \partial\Omega \setminus \Sigma.$$

  Of course, due to the incompressibility, one must have $\int_\Sigma g = 0$.
- When $g \neq 0$, this is not sufficient to determine the solution uniquely. Yudovich proposes to add the entering vorticity:

  $$\omega \ \text{ on } \ \Sigma^- := \{(t,x) \in [0,T] \times \Sigma, \ u(t,x) \cdot n(x) < 0\}.$$
- Under suitable conditions, Yudovich proves the existence and uniqueness of regular solutions to this initial-boundary system.
- See the recent paper of Noisette-Sueur (2021) removing this assumption of regularity.

Controllability problem

- Here one cannot control the fluid velocity due, for instance, to Kelvin's theorem.
- But one can try to control the solids' positions/velocities.
- The controllability problem for the solid then reads as follows: given $T > 0$ and
  - $(h_{i,0}, \theta_{i,0})$ and $(h'_{i,0}, \theta'_{i,0})$ initial positions/velocities of the solid,
  - u_0 an initial fluid velocity field (with compatibility conditions),
  - $(h_{i,1}, \theta_{i,1})$ and $(h'_{i,1}, \theta'_{i,1})$ target positions/velocities of the solid,

  can we find a control function on the boundary so that the solution starting from $((h_{i,0}), (\theta_{i,0}), (h'_{i,0}), (\theta'_{i,0}), u_0)$ at initial time satisfies at time T:

$$(h_i(T), \theta_i(T), h'_i(T), \theta'_i(T)) = (h_{i,1}, \theta_{i,1}, h'_{i,1}, \theta'_{i,1}) \text{ for } i = 1 \dots N?$$

Controlling *trajectories*

- One can actually get a stronger result, by controlling the solids' trajectories during the whole time interval $[0, T]$.
- The problem now reads as follows: given $T > 0$ and trajectories $t \mapsto (h_i^\star, \theta_i^\star)(t)$ avoiding contacts, can we find a control function on the boundary so that the corresponding solution satisfies:

$$(h_i, \theta_i) = (h_i^\star, \theta_i^\star) \text{ in } [0, T] \text{ for } i = 1, \dots, N?$$

Our framework

- The second control of Yudovich will be chosen as

$$\omega = 0 \text{ on } \Sigma^- := \{(t,x) \in [0,T] \times \Sigma,\ u(t,x) \cdot n(x) < 0\}.$$

  Hence the control relies on the normal part of the velocity.
- A consequence is that if we start from an irrotational flow, the irrotational character is kept all over time.
- We will work in vorticity formulation (to allow a clear separation of control and state)
- Since we consider only solutions without contacts, we parameterize $(\mathcal{S}_i)_{i=1\dots N}$ by $q = ((h_i, \theta_i))$ and introduce

$$\mathcal{Q} = \{q = ((h_i),(\theta_i)) \in \mathbb{R}^{3N} \ / \ \mathcal{S}_i(q) \subset \Omega,\ d(\mathcal{S}_i(q), \partial\Omega) > 0 \text{ and } d(\mathcal{S}_i(q), \mathcal{S}_j(q)) > 0\}.$$

$$\mathcal{Q}_\delta = \{q = ((h_i),(\theta_i)) \in \mathbb{R}^{3N} \ / \ \mathcal{S}_i(q) \subset \Omega,\ d(\mathcal{S}(q), \partial\Omega) > \delta \text{ and } d(\mathcal{S}_i(q), \mathcal{S}_j(q)) > \delta\}.$$

Main result

Theorem (G.-Kolumban-Sueur)

Suppose that no solid is a disk. Let $T > 0$.

1. *(Existence) For any given trajectory q in $C^2([0,T];\mathcal{Q})$, any initial vorticity ω_0 in $L^\infty(\mathcal{F}(q(0)))$, any γ in $\mathbb{R}^N$, there exists a velocity field u with $u(t,\cdot) \in LL(\mathcal{F}(t))$, with curl $u(0,\cdot) = \omega_0$ and the circulation of $u(0,\cdot)$ along $\partial\mathcal{S}_i(0)$ is γ_i, such that*

   $$(q, u) \text{ satisfies the system.}^1$$

2. *(Control form) Given $\delta > 0$ and $r > 0$, one can choose g in feedback form*

   $$g(t) = \mathscr{C}(q(t), q'(t), q''(t), \gamma, \text{curl } u(t,\cdot)),$$

   valid provided that $q \in C^2([0,T];\mathcal{Q}_\delta)$ and $\|\omega_0\|_\infty \leq r$, $|\gamma| \leq r$. Moreover, $\mathscr{C}$ has a finite-dimensional range.

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[1] (Euler for the fluid, Newton for the solids, the interface condition on the boundaries of the solids, the boundary condition on the normal velocity with this control.)

Main result, continuation

3. (Uniqueness of solutions concerning the the solids' trajectories) Let $(\tilde{q}, \tilde{u})$ another solution of the system of the same regularity with the same feedback control

$$g(t) = \mathscr{C}(q(t), q'(t), q''(t), \tilde{\gamma}, \text{curl } \tilde{u}(t, \cdot)),$$

but possibly different initial vorticity and circulations, still satisfying $\| \text{curl} \, (\tilde{u}(0, \cdot))\|_\infty \leq r$ and $|\tilde{\gamma}| \leq r$.

Then $\tilde{q} = q$ on $[0, T]$.

Remark. When $\mathcal{S}_i$ is a disk, then

$$\mathcal{J}_i \, \vartheta_i''(t) = \int_{\partial \mathcal{S}_i(t)} p \, (x - h_i(t))^\perp \cdot n \; ds = 0!$$

Connected results

This is the first result (to our knowledge) on the boundary controllability of the Euler/solid system, but there are local exact controllability results for the Navier-Stokes/solid system:

- Boulakia-Osses,
- Imanuvilov-Takahashi,
- Boulakia-Guerrero.

These results rely on parabolic techniques for the control of PDEs (Carleman estimates).

Two elements of proof.

Let us briefly describe two element of proofs:

1. Reformulation the problem (almost) into an ODE,
2. How to rely on the quadratic effect of the control.

1. Reformulating the problem into an ODE

Before considering the general case, we get back to a result concerning the potential case ($\omega = 0$, $\gamma = 0$) and without control ($u \cdot n = 0$ on $\partial\Omega$). We present the one-solid case:

Theorem (Munnier)

In the potential case without control, the system is equivalent, for a certain field $M : Q \to S^{++}(\mathbb{R}^3)$ of symmetric positive-definite matrices, to the equation in $q = (h, \theta)$:

$$M(q)q'' + \langle \hat{\Gamma}(q), q', q' \rangle = 0,$$

where $\hat{\Gamma}$ is a bilinear symmetric mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, with

$$\hat{\Gamma}_{ij}^k := \frac{1}{2} \left(\frac{\partial M_{ik}}{\partial q_j} + \frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ij}}{\partial q_k} \right).$$

Remark. It is the geodesics equation with respect to the Riemannian metric given by $M(q)$. Up to inversion by $M(q)$, the $\hat{\Gamma}_{ij}^k$ are the Christoffel symbols associated to this metric.

A very light idea of proof: what is $M(q)$?

- One introduces Kirchhoff's potentials $\Phi := (\Phi_1, \Phi_2, \Phi_3)$: $\Phi_i = \Phi_i(q, x)$ where:

$$\Delta_x \Phi_i(q, \cdot) = 0 \ \text{ in } \ \mathcal{F}(q),$$

$$\partial_n \Phi_i = 0 \text{ on } \partial\Omega, \quad \text{and} \quad \partial_n \Phi_i = \begin{cases} n_i \quad (i = 1, 2) \\ (x - h)^\perp \cdot n \quad (i = 3) \end{cases} \text{ on } \ \partial\mathcal{S}(q).$$

- Since the flow is potential, we have then

$$u = h_1' \nabla\Phi_1 + h_2' \nabla\Phi_2 + \theta' \nabla\Phi_3.$$

The solid equation becomes

$$\begin{pmatrix} m \operatorname{Id}_2 & 0 \\ 0 & \mathcal{J} \end{pmatrix} \begin{pmatrix} h' \\ \theta' \end{pmatrix}' = \left(\int_{\partial\mathcal{S}(t)} p \partial_n \Phi_i \, dx \right)_{i=1,2,3} = \left(\int_{\mathcal{F}(t)} \nabla p \cdot \nabla\Phi_i \, dx \right)$$

- Continuing the computation we have

$$\begin{pmatrix} m\,\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{pmatrix} \begin{pmatrix} h' \\ \theta' \end{pmatrix}' = \left(\int_{\mathcal{F}(t)} (-\partial_t u - (u \cdot \nabla) u)) \cdot \nabla \Phi_i \, dx \right)$$
$$= -\left(\int_{\mathcal{F}(t)} (h_1'' \nabla \Phi_1 + h_2'' \nabla \Phi_2 + \theta'' \nabla \Phi_3) \cdot \nabla \Phi_i \, dx \right) \underbrace{+ \text{ shape derivative terms } + \text{ terms in } (u \cdot \nabla) u}_{\text{terms containing no second derivative of } q}$$

- It follows that the equation can be put in the form

$$\mathcal{M} \begin{pmatrix} h' \\ \vartheta' \end{pmatrix}' = \text{ quadratic terms in } q'$$

where

$$\mathcal{M} := \begin{pmatrix} m\,\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{pmatrix} + \underbrace{\left(\int_{\mathcal{F}(t)} \nabla \Phi_i \cdot \nabla \Phi_j \, dx \right)_{i,j=1,2,3}}_{=:\mathcal{M}_a}.$$

The matrix $\mathcal{M}_a$ is a matrix of added inertia, expressing how the fluid opposes the movement of the solid. It is positive, and even positive definite when $\mathcal{S}_0$ is not a disk, as a Gram matrix of independent functions.

In our case: with circulations, vorticity and control

In our case:

- Kirchhoff potentials have to be introduced for each solid,
- three terms have to be added in the description of u to take the circulations, the vorticity and the control into account, which result in an important number of terms.

In particular we introduce α by

$$\Delta_x \alpha(q,\cdot) = 0 \;\text{ in }\; \mathcal{F}(q),$$
$$\partial_n \alpha = 0 \text{ on } \partial\mathcal{F}(t) \setminus \Sigma^-(t,\cdot), \quad \partial_n \alpha = g(t,\cdot) \text{ on } \Sigma^-(t,\cdot), \quad .$$

Theorem

The solid equation can be written in the form:

$$M(q)q'' = -\int_{\partial\mathcal{S}(q)} \left(\partial_t \alpha + \frac{|\nabla \alpha|^2}{2} \right) \partial_n \Phi \, ds + \tilde{\mathfrak{L}}(q,q',\gamma,\omega)[g] + \mathfrak{F}(q,q',\gamma,\omega),$$

where $M(q)$ is the $3N \times 3N$ total mass matrix and $\tilde{\mathfrak{L}}$ is linear in g.

Corollary

Under the additional condition on g that

$$\int_{\partial \mathcal{S}_i} \alpha \partial_n \Phi_{i,j} \, ds = 0, i = 1, \ldots, N, \quad j = 1, 2, 3,$$

the equation can be written as

$$\mathfrak{Q}(q)[g] + \mathfrak{L}(q, q', \gamma, \omega)[g] = \mathfrak{F}(q, q', \gamma, \omega) + M(q) q''$$

with $\mathfrak{L}$ linear in g and

$$\mathfrak{Q}(q)[g] := \frac{1}{2} \left(\int_{\partial \mathcal{S}_i(q)} |\nabla \alpha|^2 \, \partial_n \Phi_{i,j}(q, \cdot) \, \mathrm{d}s \right)_{i=1 \ldots N, \ j=1,2,3}.$$

2. Relying on the quadratic effect of the control

- We construct controls g that satisfy the additional condition

$$\int_{\partial \mathcal{S}_i} \alpha \partial_n \Phi_{i,j} \, ds = 0, i = 1, \dots, N, \quad j = 1, 2, 3,$$

- When considering the final time controllability only, one can use Coron's scaling argument:
  - rescale in time,
  - act strong and fast,
  - use that for large and well-chosen g, the quadratic term $\mathfrak{Q}$ is dominant with respect to the linear term $\mathfrak{L}$; hence one can invert the equation and find g.

General strategy

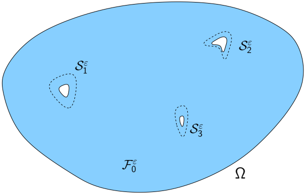
- Here we cannot rescale in time, but still manage to invert relying on the quadratic part (this is again reminiscent of Coron's ideas).
- Roughly speaking, we look for a solution X of
$$Q(X) + L(X) = Y,$$
where Q is quadratic and L is linear (actually depending on a lot of parameters).
- The idea is to find a point $\overline{X}$ such that
$$\|\overline{X}\| = 1, \quad Q(\overline{X}) = 0, \ DQ(\overline{X}) \text{ is right-invertible.}$$
- Then one can find a solution by using a similar scaling argument (to work with $Q(X) + \varepsilon L(X) = Y$) and linearizing around $\overline{X}$.
- In the end we find a solution close to $\lambda\overline{X}$, λ large.

Part III. Controlling vortices in a perfect fluid

- A natural problem would be, rather than controlling from the boundary, to use one of the trajectory of one of the solids as a control to influence the others.
- One can see the question of controlling point vortices by means of one of them as a simplified version of this problem.
- We have indeed proved in a series of paper with F. Sueur, C. Lacave and A. Munnier that in the above system, if one considers solids whose size and mass converge to zero

$$\mathcal{S}^{\varepsilon}_{i,0} := h_{i,0} + \varepsilon_i (\mathcal{S}_{i,0} - h_{i,0}),$$

while the velocity circulation around them is fixed, one obtains in the limit the point vortex system.



Point vortex system

- The point vortex system was originally introduced as a simplified model for the Euler equation (Helmholtz, Kirchhoff, Kelvin and Poincaré), where the vorticity is concentrated in a finite number of points.
- It mainly consists in considering the vorticity (transport) equation

  $$\partial_t \omega + \operatorname{div}(u\omega) = 0,$$

  where the vorticity is a sum of Dirac masses

  $$\omega(t) = \sum_{k=i}^{N} \gamma_i \delta_{x_i(t)},$$

  and u is obtained by means of the *div/curl* system.
  But one has to remove the self-interaction of vortices, that is, the i-th vortex is transported by the flow generated by the others.
- There are rigourous proofs of the limit of the solutions of Euler equations to the point vortex system (see e.g. Marchioro & Pulvirenti's book)

Point vortex system

- This results into the following system of ordinary differential equations:

$$\frac{dx_i}{dt}(t) = \sum_{j=1,\, j\neq i}^{N} \frac{\gamma_j}{2\pi} \frac{(x_i(t)-x_j(t))^{\perp}}{|x_i(t)-x_j(t)|^2}, \quad i = 1, \ldots, N.$$

- This naturally has local-in-time regular solutions, but these may blow up when two vortices meet.
- Going back to our control problem, we are led to consider:

$$\frac{dx_i}{dt}(t) = \sum_{j=1,\, j\neq i}^{N} \frac{\gamma_j}{2\pi} \frac{(x_i(t)-x_j(t))^{\perp}}{|x_i(t)-x_j(t)|^2} + \frac{\gamma^c}{2\pi} \frac{(x_i(t)-y(t))^{\perp}}{|x_i(t)-y(t)|^2}, \ i = 1, \ldots, N,$$

where we have N point vortices $x_1, \ldots, x_N$ and a control vortex y, whose trajectory we can choose in order to influence the others.

Main result

Our main result is as follows.

Theorem (Dorsz-G.)

Suppose $\gamma^c \neq 0$. The point vortex system is exactly controllable in arbitrary time by means of a single control vortex.

More precisely, given $T > 0$, $(x_{1,0}, \ldots, x_{N,0}, y_0) \in (\mathbb{R}^2)^{N+1}$, $(x_{1f}, \ldots, x_{Nf}, y_f) \in (\mathbb{R}^2)^{N+1}$, there exists $y \in C^\infty([0,T];\mathbb{R}^2)$ satisfying $y(0) = y_0$, $y(T) = y_f$, and such that the corresponding solution of the system is defined in $[0,T]$ and satisfies

$$(x_1(T), \ldots, x_N(T)) = (x_{1,f}, \ldots, x_{N,f}).$$

Rough ideas

The principles of the proof are fairly simple.

1. When $N = 1$, that is, when one controls a single vortex by mean of another:
$$\frac{dx_1}{dt}(t) = \frac{\gamma^c}{2\pi}\frac{(x_1(t) - y(t))^\perp}{|x_1(t) - y(t)|^2},$$
the controllability is immediate, since one can invert the right-hand side to generate any $\dot{x}_1$.

Rough ideas, 2

2. When $N > 1$, and if we use also N control vortices:

$$\frac{dx_i}{dt}(t) = \sum_{j=1,\, j\neq i}^{N} \frac{\gamma_j}{2\pi} \frac{(x_i(t) - x_j(t)^{\perp}}{|x_i(t) - x_j(t)|^2} + \sum_{j=1}^{N} \frac{\gamma_j^c}{2\pi} \frac{(x_i(t) - y_j(t))^{\perp}}{|x_i(t) - y_j(t)|^2},$$

one can also obtain a controllability result: one can make each vortex x_i follow approximately a given trajectory, provided that these trajectories are travelled sufficiently fast.

Indeed in that case, we let the i-th control vortex y_i approach the i-th controlled vortex x_i. The interaction between y_i and x_i is then strong and dominant, so the N controls/N controlled system is well approximated by N independent 1 control/1 controlled systems.

Rough ideas, 3

3. Going back to controlling $N > 1$ vortices by means of a single vortex we use ideas of Filippov's convex integration: the single vortex travels fast between the positions of N virtual vortices computed in the previous step.

4. One has to prove that this mimics efficiently the action of N vortices. This allows to control approximately the behaviour of the N vortices.
5. It remains to benefit from the finite-dimensionality of the objective to get the exact controllability by means of a Brouwer-type argument.

That's all!

Happy (belated) birthday Marius!