A unified approach of obstructions to small-time local controllability for scalar-input systems

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joint work with Jérémy Leborgne and Frédéric Marbach

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Marius : thanks for...

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- and some oenological break through.

Long-term objective : NSC for STLC of affine systems

$\frac{dx}{dt} = f_0(x) + u(t)f_1(x) \qquad x(t) \in \mathbb{R}^d,\ u(t) \in \mathbb{R},\ f_0, f_1 \in C^\omega(\mathbb{R}^d, \mathbb{R}^d)$

We assume $f_0(0) = 0$, i.e. $(x, u) = (0, 0)$ is an equilibrium trajectory.

Definition (Small Time Local Controllability)

The system is *STLC* when

$$\forall T, \eta > 0, \quad \exists \delta > 0, \quad \forall x_0, x_f \in B_{\mathbb{R}^d}(0, \delta), \quad \exists u \in L^\infty((0, T), \mathbb{R}),$$

$$\|u\|_\infty \leqslant \eta, \qquad x(T; u, x_0) = x_f.$$

Notion invariant by diffeomorphism ψ of $\mathbb{R}^d$ st $\psi(0) = 0$:
$y(t) := \psi(x(t))$ solves $\frac{dy}{dt} = g_0(y) + u(t)g_1(y)$ where $g_i = \psi' \circ f_i \circ \psi^{-1}$
Both systems, on x and y, have the same STLC properties.

Long-term objective : NSC on Lie brackets for STLC

Definition (Lie bracket)

Let f and g be smooth vector fields on $\mathbb{R}^d$. The Lie bracket $[f,g]$ is the smooth vector field defined by : $[f,g](x) := g'(x)f(x) - f'(x)g(x)$.
By induction on $k \in \mathbb{N}$: $\quad \mathrm{Ad}_f^0(g) = g\,, \qquad \mathrm{Ad}_f^{k+1}(g) = \left[f, \mathrm{Ad}_f^k(g)\right].$

With differential operators : $[f,g].\nabla = (f\cdot\nabla)(g\cdot\nabla) - (g\cdot\nabla)(f\cdot\nabla)$

[Krener 1973] : diffeomorphic systems $\Leftrightarrow$ isomorphic brackets at 0
$\dot{x} = f_0(x) + uf_1(x)$ and $y(t) = \psi(x(t))$ implies $\dot{y}(t) = g_0(y) + ug_1(y)$ where $g_i = \psi' \circ f_i \circ \psi^{-1}$ thus $[g_0, g_1] = \psi' \circ [f_0, f_1] \circ \psi^{-1}$ and $[g_0, g_1](0) = \psi'(0).[f_0, f_1](0)$

$\Rightarrow$ **The entire information about STLC of $\dot{x} = f_0(x) + uf_1(x)$ is contained in the evaluation at 0 of the Lie brackets of f_0 and f_1.**

An emblematic example of NC for STLC

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad x(t) \in \mathbb{R}^d,\ u(t) \in \mathbb{R},\ f_0, f_1 \in C^\omega,\ f_0(0) = 0$$

Theorem (Sussmann 1983 : $\ell = 1$ / Stefani 1986 : $\forall \ell$)

$$STLC \qquad \Rightarrow \qquad \forall \ell \in \mathbb{N}^*, \quad \mathrm{Ad}_{f_1}^{2\ell}(f_0)(0) \in \mathcal{S}_{[1,2\ell-1]}(0).$$

$$\mathcal{S}_{[1,2\ell-1]} := \mathrm{Span}\{\text{iterated brackets of } f_0 \text{ and } f_1, \text{ involving } f_1 \text{ at most } (2\ell-1) \text{ times}\}$$

A typical example is :

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^{2\ell} + x_1^{2\ell+1} \end{cases} \qquad \Rightarrow \qquad \begin{array}{l} \bullet\, f_1 \equiv e_1, \\ \bullet\, \mathrm{Ad}_{f_1}^{2\ell}(f_0)(0) \,/\!/\, \mathrm{Ad}_{f_1}^{2\ell+1}(f_0)(0) \,/\!/\, e_2 \\ \bullet\, \text{If } (x_1, x_2)(0) = 0 \text{ then} \\ x_2(T) \geq (1 - T\|u\|_\infty) \int_0^T x_1^{2\ell} > 0 \end{array}$$

"Bad Lie bracket" : If $\mathrm{Ad}_{f_1}^{2\ell}(f_0)(0)$ is not compensated by appropriate brackets, then it generates a drift in the dynamics.

Goal and content of this talk

$$\dot{x} = f_0(x) + u(t)f_1(x) \qquad x(t) \in \mathbb{R}^d,\ u(t) \in \mathbb{R},\ f_0, f_1 \in C^\omega,\ f_0(0) = 0$$

Objective : A unified approach to conjecture and prove necessary conditions for STLC based on
- a new representation formula of the state $x(t; u)$,
- a new basis $\mathcal{B}^\star$ of the free Lie algebra $\mathcal{L}(\{X_0, X_1\})$.

1. A new Magnus-type expansion "in interaction picture"
2. Sussmann's infinite product
3. A new Hall basis $\mathcal{B}^\star$ of $\mathcal{L}(\{X_0, X_1\})$
4. New necessary conditions for STLC

The formal differential equation : an important tool

$X = \{X_0, X_1\}$: non commutative indeterminates
$\mathcal{A}_n(X) = \text{Span}\{\text{monomials of degree } n\}$ **ex :** $X_0^{n-2} X_1 X_0$
$\widehat{\mathcal{A}}(X)$: algebra of formal series **ex :** $-\ln(1 - X_1) = \sum \frac{X_1^n}{n}$, $\sum e^{n^3} X_0^n$

Definition (Solution to a formal ODE)

Let $u \in L^1(\mathbb{R}_+, \mathbb{R})$. The solution to the formal ODE

$$\begin{cases} \dot{x}(t) = x(t)\big(X_0 + u(t)X_1\big) \\ x(0) = 1 \end{cases}$$

is the formal-series valued function $x : \mathbb{R}_+ \to \widehat{\mathcal{A}}(X)$, whose homogeneous components $x_n : \mathbb{R}_+ \to \mathcal{A}_n(X)$ satisfy, $\forall t \geq 0$, $x_0(t) = 1$ and $\forall n \in \mathbb{N}^*$,

$$x_n(t) = \int_0^t x_{n-1}(\tau)(X_0 + u(\tau)X_1)\,\mathrm{d}\tau.$$

If $u = 0$ then $x(t) = \sum \frac{t^n}{n!} X_0^n = e^{tX_0}$.
If $u(t) = t$, then $x_1(t) = tX_0 + \frac{t^2}{2} X_1$, etc.

Chen-Fliess expansion for the formal differential equation

By iterating the integral formula $x_n(t) = \int_0^t x_{n-1}(\tau)(X_0 + u(\tau)X_1)\,\mathrm{d}\tau$ we obtain

$$x(t) = \sum_{\sigma \in \cup_{k\in\mathbb{N}}\{0,1\}^k} \left(\int_0^t u_\sigma\right) X_{\sigma_1} \dots X_{\sigma_k},$$

where, $u_0 = 1$ and $u_1 = u$ and for $\sigma = (\sigma_1, \dots, \sigma_k) \in \{0,1\}^k$,

$$\int_0^t u_\sigma := \int_{0<\tau_1<\dots<\tau_k<t} u_{\sigma_1}(\tau_1)\cdots u_{\sigma_k}(\tau_k)\,\mathrm{d}\tau,$$

Kuo Tsai Chen (1923-1987), *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula, Annals of Maths, 1957*
Michel Fliess, *Fonctionelles causales nonlinéaires et indéterminées noncommutatives*, Bull. SMF, 1981

Why the formal DE is important : the linearization principle

One transforms the **nonlinear ODE in** $\mathbb{R}^d$ **:** $\dot{x} = f_0(x) + u(t)f_1(x)$, into a **linear equation in Op**$(C^\infty(\mathbb{R}^d, \mathbb{R}))$ **:** $\frac{\mathrm{d}}{\mathrm{d}t}L(t) = L(t)(\underbrace{f_0 \cdot \nabla}_{X_0} + u(t)\underbrace{f_1 \cdot \nabla}_{X_1})$

on which one can use the algebra developed for the formal ODE.

Consider the zero-order operator

$$\begin{array}{rccc} L(t): & C^\infty(\mathbb{R}^d, \mathbb{R}) & \to & C^\infty(\mathbb{R}^d, \mathbb{R}) \\ & \varphi & \mapsto & (p \mapsto \varphi(x(t; u, p))) \end{array}$$

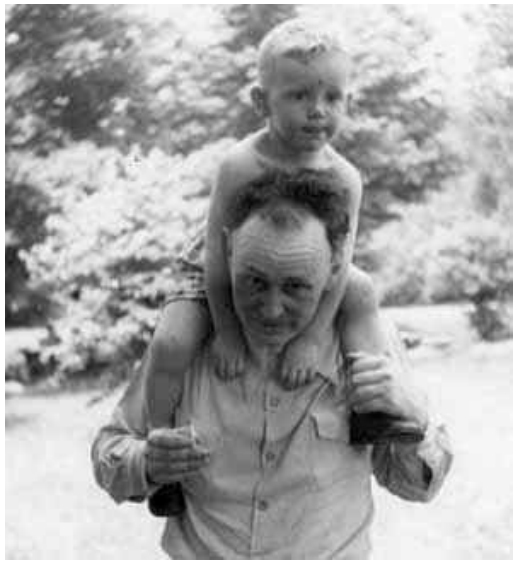
Then, $\forall \varphi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $\forall p \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(L(t)\varphi\big)(p) &= D\varphi\Big(x(t; f, p)\Big).(f_0 + u(t)f_1)(x(t; f, p)) \\ &= \Big(L(t)(f_0 \cdot \nabla + u(t)f_1 \cdot \nabla)\varphi\Big)(p) \end{aligned}$$

So, in this weak sense, $\frac{\mathrm{d}}{\mathrm{d}t}L(t) = L(t)(f_0 \cdot \nabla + u(t)f_1 \cdot \nabla)$.

A new Magnus-type expansion "in interaction picture"

Wilhelm Magnus (1907-1990)



The classical Magnus expansion

$$\dot{x}(t) = x(t)(X_0 + u(t)X_1) \qquad\qquad x(0) = 1$$

Exponential on formal series : For $a \in \widehat{\mathcal{A}}(X)$, $e^a := \sum_{m\geq 0} \frac{a^m}{m!} \in \widehat{\mathcal{A}}(X)$.

Theorem (Magnus 1954)

$x(t) = e^{Z(t,u)}$ *where* $Z(t,u)$ *is a formal series* ***of Lie brackets of*** X_0, X_1

$$Z(t,u) = \sum_{r=1}^{\infty} \frac{1}{r} \sum_{m=1}^{r} \frac{(-1)^{m-1}}{m} \sum_{\mathbf{r}\in\mathbb{N}_r^m} \int_{\tau\in\mathcal{T}_{\mathbf{r}}(t)} [\cdots[F(\tau_r), F(\tau_{r-1})], \ldots F(\tau_1)]\, \mathrm{d}\tau$$

where $F(\tau) = X_0 + u(\tau)X_1$ *and* $\mathcal{T}_{\mathbf{r}}(t)$ *is a product of pyramidal domains.*

Classical proof, for instance [Strichartz 1987]

- $Z(t,u) :=$ log(Chen Fliess expansion) $\in \widehat{\mathcal{L}}(X)$ by Friedrichs' criterion
- Dynkin : if $a \in \mathcal{A}_r(X)$ then $a \in \mathcal{L}(X)$ iff $a = r *$ (left bracketting of a)

If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$ then $Z(t,u) = \sum_{b\in\mathcal{B}} \zeta_b(t,u) b$.
ζ_b : **"coordinates of the first kind associated with** $\mathcal{B}$**"**, $\sim C^{|b|}$

Drawback for ODEs, the infinite segment that would need to be summed for an error estimate in u does not converge.

New : Magnus expansion "in interaction picture"

$$\dot{x}(t) = x(t)(X_0 + u(t)X_1) \qquad\qquad x(0) = 1$$

We isolate the role of the autonomous drift X_0 from the role of the time-varying perturbation $u(t)X_1$.

Theorem (KB, FM, JLB 2020)

$x(t) = e^{tX_0} e^{\mathcal{Z}(t,u)}$ *where* $\mathcal{Z}(t,u)$ *is a formal series of* ***Lie brackets of*** X_0, X_1

$$\mathcal{Z}(t,u) = \sum \frac{(-1)^{m-1}}{mr} \int_{\tau \in \mathcal{T}_{\mathbf{r}}(t)} \frac{(\tau_r - t)^{k_r}}{k_r!} \cdots \frac{(\tau_1 - t)^{k_1}}{k_1!} u_{i_r}(\tau_r) \cdots u_{i_1}(\tau_1) \, \mathrm{d}\tau$$
$$[\cdots[\mathrm{Ad}_{X_0}^{k_r}(X_{i_r}), \mathrm{Ad}_{X_0}^{k_{r-1}}(X_{i_{r-1}})], \ldots \mathrm{Ad}_{X_0}^{k_1}(X_{i_1})]$$

where $u_0 = 1$, $u_1 = u$ *and the sum is taken over* $r \in [\![1, \infty]\!]$, $m \in [\![1, r]\!]$, $\mathbf{r} \in \mathbb{N}_r^m$, $k_1, \ldots, k_r \in \mathbb{N}$ *and* $i_1, \ldots, i_r \in \{0, 1\}$.

If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$ then $\mathcal{Z}(t,u) = \sum_{b \in \mathcal{B}} \eta_b(t,u) b$.

η_b : "**coordinates of the pseudo-first kind**", $\sim \frac{C^{|b|}}{|b|!}$ on a Hall basis homogeneous with degree $n_1(b)$ wrt u, $|b|$ wrt t, causal.

The new expansion for ODEs : error estimate in u

$\dot{x} = f_0(x) + u(t)f_1(x)$ $\qquad x(0) = p$ $\qquad f_0, f_1 \in \mathcal{C}^\omega, f_0(0) = 0$

Theorem (KB, FM, JLB 2020)

For $M \in \mathbb{N}$, $x(t;u,p) = e^{\mathcal{Z}_M(t,u)} e^{tf_0}(p) + O(\|u_1\|_{L^{M+1}}^{M+1})$ *when* $(T, \|u_1\|_\infty) \to 0$, *where $\mathcal{Z}_M(t,u)$ is the analytic vector field*

$$\mathcal{Z}_M(t,u) = \sum \frac{(-1)^{m-1}}{rm} \int_{\mathcal{T}_{\mathbf{r}}(t)} \frac{(\tau_1 - t)^{k_1}}{k_1!} \cdots \frac{(\tau_r - t)^{k_r}}{k_r!} u(\tau_r) \dots u(\tau_1) \, \mathrm{d}\tau$$
$$\left[\cdots \left[\mathrm{Ad}_{f_0}^{k_r}(f_1), \mathrm{Ad}_{f_0}^{k_r - 1}(f_1) \right], \dots, \mathrm{Ad}_{f_0}^{k_1}(f_1) \right]$$

*where the sum is taken over $r \in [\![1, M]\!]$, $m \in [\![1, r]\!]$, $\mathbf{r} \in \mathbb{N}_r^m$, $k_1, \dots, k_r \in \mathbb{N}$ and the **series** converges absolutely in $\mathcal{C}^\omega$.*
If $\mathcal{B}$ is Hall basis of $\mathcal{L}(X)$ then $\mathcal{Z}_M(t,u) = \sum_{b \in \mathcal{B} \cap S_{[1,M]}} \eta_b(t,u) f_b$.

<u>Notation</u> *: evaluated brackets, if $b = [X_0, [X_0, X_1]]$ then $f_b := [f_0, [f_0, f_1]]$.*
Using $e^Z(0) = Z(0) + O(|Z|_{C^1} |Z(0)|)$ we get
an approximate representation of the state : when $(T, \|u_1\|_\infty) \to 0$

$$x(T;u,0) = \mathcal{Z}_M(T,u)(0) + O\left(\|u_1\|_{L^{M+1}}^{M+1}\right) + o\left(|x(T;u,0)|\right)$$

A unified approach for obstructions ... in an ideal world

$\dot{x} = f_0(x) + u(t)f_1(x) \qquad x(0) = 0 \qquad f_0, f_1 \in \mathcal{C}^\omega, f_0(0) = 0$

$$x(T;u) = \sum_{b \in \mathcal{B} \cap S_{[\mathbf{1},M]}} \eta_b(T,u) f_b(0) + O\left(\|u_1\|_{L^{M+\mathbf{1}}}^{M+\mathbf{1}}\right) + o\left(|x(T;u)|\right)$$

If

- for some $\overline{b} \in \mathcal{B}$, the functionnal $\eta_{\overline{b}}(T,.)$ is signed for T small,
- for some $M \in \mathbb{N}$, $\int_0^T |u_1|^{M+1} = o(\eta_{\overline{b}}(T,u))$ when $(T, \|u\|_\infty) \to 0$,
- for some $E \subset \mathcal{B} \cap S_{[1,M]} \setminus \{\overline{b}\}$, $\sum_{b \in E} \eta_b(T,u) f_b(0) = o(\eta_{\overline{b}}(T,u))$,

then a necessary condition for STLC is

$$f_{\overline{b}}(0) \in \mathrm{Span}\left\{f_b(0); b \in \mathcal{B} \cap S_{[1,M]} \setminus (E \cup \{\overline{b}\})\right\}$$

because otherwise, $x(T;u)$ drifts along $f_{\overline{b}}(0)$.

Pb : The coordinates of the pseudo-first kind η_b are difficult to compute.

Sussmann's infinite product & computation of the coordinates of the pseudo-first kind

Hall basis of $\mathcal{L}(X)$ and coordinates of the second kind

A **Hall set** over X is a totally ordered subset $\mathcal{B} \subset \mathrm{Br}(X)$ such that

- $X_0, X_1 \subset \mathcal{B}$
- for all $b_1, b_2 \in \mathrm{Br}(X)$, $(b_1, b_2) \in \mathcal{B}$ iff $b_1, b_2 \in \mathcal{B}$, $b_1 < b_2$ and either $b_2 \in X$ or $b_2 = (b_2', b_2'')$ where $b_2' \leq b_1$,
- for all $b_1, b_2 \in \mathcal{B}$ such that $(b_1, b_2) \in \mathcal{B}$ then $b_1 < (b_1, b_2)$.

Theorem (Viennot 1978)

The evaluation of a Hall set is a basis of $\mathcal{L}(X)$.

The **coordinates of the second kind** associated with $\mathcal{B}$ is the family of functionals $(\xi_b(t,u))_{b\in\mathcal{B}}$ defined by induction by $\xi_{X_0}(t,u) = t$, $\xi_{X_1}(t,u) = \int_0^t u$ and

$$\xi_{\mathrm{ad}_a^m(b)}(t,u) = \int_0^t \frac{\xi_a(\tau)^m}{m!}\dot{\xi}_b(\tau)d\tau$$

Infinite product for the formal equation

$$\dot{x}(t) = x(t)(X_0 + u(t)X_1) \qquad\qquad x(0) = 1$$

Theorem (Sussmann 1986)

Let $\mathcal{B} \subset \mathrm{Br}(X)$ be a Hall set. Then

$$x(t) = \overleftarrow{\prod_{b \in \mathcal{B}}} e^{\xi_b(t;u)b}$$

Proof : Eliminate X_1 by looking for $x(t)$ of the form $y(t)e^{\xi_{X_1}(t)X_1}$.

$$\begin{aligned}
\dot{y}(t) &= x(t)(X_0 + u(t)X_1)e^{-\xi_{X_1}(t)X_1} + x(t)(-\dot{\xi}_{X_1}(t)X_1)e^{-\xi_{X_1}(t)X_1} \\
&= y(t)e^{\xi_{X_1}(t)X_1}X_0 e^{-\xi_{X_1}(t)X_1} \qquad \text{because } \dot{\xi}_{X_1}(t) = u(t) \\
&= y(t)\sum_{m\in\mathbb{N}} \underbrace{\frac{\xi_{X_1}(t)^m}{m!} \cdot \dot{\xi}_{X_0}(t)}_{=\dot{\xi}_{\mathrm{ad}^m_{X_1}(X_0)}(t)} \mathrm{ad}^m_{X_1}(X_0) \qquad \rightarrow \text{does not involve } X_1
\end{aligned}$$

Lazard elimination

Continue the process : If at some step

$$\dot{y}(t) = \sum_{b \in H_y} \dot{\xi}_b(t) b,$$

choose a bracket $a \in H_y$ and look for $y(t)$ of the form $z(t)e^{\xi_a(t)a}$. Then

$$\dot{z}(t) = z(t) \sum_{m \in \mathbb{N}} \sum_{b \in H_y \setminus \{a\}} \underbrace{\frac{(\xi_a(t))^m}{m!} \dot{\xi}_b(t)}_{\dot{\xi}_{\mathrm{ad}_a^m(b)}(t)} \mathrm{ad}_a^m(b)$$

There are many more (new) brackets, but a has been eliminated.
And we are back to the initial point.

Coord. of the pseudo-1st kind wrt coord. of the 2nd kind

Formal equation : $\dot{x}(t) = x(t)(X_0 + u(t)X_1) \qquad x(0) = 1$
Let $\mathcal{B}$ be a Hall set with X_0 maximal. Then $\mathcal{Z}(t,u) = \sum_{b\in\mathcal{B}} \eta_b(t) b$ and

$$x(t) = e^{X_0 t} e^{\mathcal{Z}(t,u)} = \overleftarrow{\prod_{b\in\mathcal{B}}} e^{\xi_b(t;u) b}$$

thus $\eta_b = \xi_b + \sum$ polynomial of$\{\xi_{b_i}\}$ where $|b_i| < |b|$ and $n_1(b_i) < n_1(b)$.

For ODEs : $\dot{x} = f_0(x) + u(t) f_1(x) \qquad x(0) = 0 \qquad f_0, f_1 \in \mathcal{C}^\omega, f_0(0) = 0$

$$x(T;u) = \sum_{b\in\mathcal{B}\cap S_{[\mathbf{1},M]}} \xi_b(T,u) f_b(0) + \text{pollution} + O\left(\int_0^T |u_1|^{M+1}\right) + o\left(|x(T;u)|\right)$$

Unfortunately, in general, the pollution is not negligible.

A unified approach for obstructions ... in the real world

$$x(T;u) = \sum_{b \in \mathcal{B} \cap S_{[\mathbf{1},M]}} \xi_b(T,u) f_b(0) + \text{pollution} + O\left(\int_0^T |u_1|^{M+1}\right) + o\left(|x(T;u)|\right)$$

If

- for some $\overline{b} \in \mathcal{B}$, the functionnal $\xi_{\overline{b}}(T,.)$ is signed,
- for some $M \in \mathbb{N}$, $\int_0^T |u_1|^{M+1} = o(\xi_{\overline{b}}(T,u))$ when $(T, \|u\|_\infty) \to 0$
- for some $E \subset \mathcal{B} \cap S_{[1,M]} \setminus \{\overline{b}\}$, $\sum_{b \in E} \xi_b(T,u) f_b(0) = o(\xi_{\overline{b}}(T,u))$,

then a necessary condition for STLC is

$$f_{\overline{b}}(0) \in \text{Span}\left\{f_b(0); b \in \mathcal{B} \cap S_{[1,M]} \setminus (E \cup \{\overline{b}\})\right\}$$

provided that, under this algebraic assumption, one may prove

$$\text{pollution} = o(\eta_{\overline{b}}(T,u)) + o\left(|x(T;u)|\right)$$

Drawback : On the known Hall basis (length-compatible, Lyndon), one does not see easily signed functionnals ξ_b.

A new Hall basis $\mathcal{B}^\star$ with nice coordinates of the second kind

A new Hall basis $\mathcal{B}^\star$

Notation : $b0^\nu := [\dots[b, X_0], \dots, X_0]$ with ν occurrences of X_0

Definition of $(\mathcal{B}^\star, <)$:

- X_0 is the maximal element of $\mathcal{B}^\star$
- if $b_1 < b_2 \neq X_0$ then $n_1(b_1) \leq n_1(b_2)$
- if $b_1 < b_2 \notin \{b_1 0^\nu ; \nu \in \mathbb{N}\}$ then $b_1 0^{\nu_1} < b_2 0^{\nu_2}$

Germ : for every $b \in \mathcal{B}^\star$ there exists a unique $b^* \in \mathcal{B}^\star$ and $\nu \in \mathbb{N}$ maximal such that $b = b^* 0^\nu$.

$$\xi_b(t, u) = \int_0^t \frac{(t-\tau)^\nu}{\nu!} \xi_{b^*}(\tau, u) d\tau$$

$\rightarrow$ Determine the germs b^* for which ξ_{b^*} is signed.

Some elements of $\mathcal{B}^\star$ and their coordinates of the 2nd kind

$\mathcal{B}^\star \cap S_1 : M_{j-1} = X_1 0^{j-1} \quad \to \quad u_j(t)$: j-th primitive of u

$\text{Germs}(\mathcal{B}^\star \cap S_2) : W_j = [M_{j-1}, M_j] \quad \to \quad \int_0^t u_j(\tau)^2 d\tau$

$\text{Germs}(\mathcal{B}^\star \cap S_3) : P_{j,k} = [M_{k-1}, W_j] \quad \to \quad \int_0^t u_j(\tau)^2 u_k(\tau) d\tau$

$\text{Germs}(\mathcal{B}^\star \cap S_4) : Q_{j,k,\ell} = [M_{\ell-1}, P_{j,k}] \quad \to \quad \int_0^t u_j(\tau)^2 u_k(\tau) u_\ell(\tau) d\tau$

$$Q^\sharp_{j,\mu,k} = [W_{j,\mu}, W_k] \quad \to \quad \int_0^t \left(\int_0^\tau \frac{(\tau-s)^\mu}{\mu!} u_j(s)^2 ds \right) u_k(\tau)^2 d\tau$$

$$Q^\flat_{j,\mu} = [W_{j,\mu}, W_{j,\mu+1}] \quad \to \quad \int_0^t \left(\int_0^\tau \frac{(\tau-s)^\mu}{\mu!} u_j(s)^2 ds \right)^2 d\tau$$

where implicitly $1 \leq j \leq k \leq \ell$.

Examples : W_j, $Q_{1,2,2}$ have signed coordinates of the second kind.

Have surjective coordinates of the second kind :
- $Q_{j,k,\ell}$ when $k < \ell$ and $k + \ell$ is odd, i.e. type (n_1 odd, n_0 odd) : $\check{u}$,
- $Q_{1,1,3}$ even if $n_0 = 4$ is odd. (cf Sussmann's cdt $S(\theta)$)

New necessary conditions for STLC

Examples of new necessary conditions for STLC

$\dot{x} = f_0(x) + u(t)f_1(x)$ $\qquad x(0) = 0$ $\qquad f_0, f_1 \in \mathcal{C}^\omega, f_0(0) = 0$

Theorem (KB FM, conjectured by Kawski 1986)

$STLC \Rightarrow \forall j \in \mathbb{N}^*, f_{W_j}(0) \in S_{[1,2j-1]}(f)(0)$ *where* $W_j = [X_1 0^{j-1}, X_1 0^j]$

Theorem

$STLC \Rightarrow f_{W_1}(0) \in S_1(f)(0)$, *[Sussmann 1983]*
$f_{W_2}(0) \in Span\left\{f_b(0); b \in S_1 \cup \{\mathrm{Ad}^3_{X_1}(X_0)0^\nu, \nu \in \mathbb{N}\}\right\}$, *[Kawski 1987]*
$f_{W_3}(0) \in Span\left\{f_b(0); b \in S_1 \cup \textit{sharp list of } \mathcal{B}^\star \cap S_{[3,5]}\right\}$. *[KB FM 2020]*

See Mégane Bournissou's poster : bilinear Schrödinger PDE
$W_3 = [X_1 0^2, X_1 0^3]$ versus $P_{1,2} = [[X_1, X_0], [X_1, [X_1, X_0]]]$.

Theorem (KB FM)

$STLC \Rightarrow f_Q(0) \in Span\{f_b(0); b \in \mathcal{B}^\star \cap S_{[1,5]} \setminus \{Q\}\}$
where $Q = Q_{1,2,2} = \mathrm{Ad}^2_{(X_1,X_0)}\mathrm{Ad}^2_{X_1}(X_0)$.

... etc