New Developments in Control Theory for Differential Equation Networks: from Trees to General graphs

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# Introduction

- Differential equation networks (DENs), or quantum graphs, are metric graphs with differential equations on edges coupled by certain vertex matching conditions.
- These models play a fundamental role in many problems of science and engineering:
- The classical problem is the problem of oscillations of the flexible structures of strings, beams, cables.
- Recently, quantum graphs were applied to description of nanostructured materials like ceramic or metallic foams, percolation networks and carbon and graphene nano-tubes.

# Graphs



# Metric graphs

Let  $\Omega(V, E)$  be a finite, connected graph,  $V = \{v_i : i \in I\}$  is the set of vertices;  $E = \{e_j : j \in J\}$  is the set of edges. By  $E(v_i)$  we denote the set of the edges incident to  $v_i : E(v_i) = \{e_j : j \in J(v_i)\}$ . The set  $\Gamma := \{v_i \in V : |J(v_i)| = 1\}$  plays the role of the graph boundary.

A graph is called a **metric graph** if every edge  $e_j \in E$  is identified with an interval  $(0, I_i)$  with a positive length  $I_i$ .

# Quantum graphs

The graph  $\Omega$  determines the Hilbert space  $\mathcal{H} := L^2(\Omega)$ . We define the space  $\mathcal{H}^1$  of continuous functions y on  $\Omega$  such that  $y_j := y|_{e_j} \in H^1(e_j) \ \forall j \in J$ . Let  $\mathcal{H}^{-1}$  be the dual space of  $\mathcal{H}^1$ . We introduce the space  $\mathcal{H}^2$  of continuous functions y on  $\Omega$  such that  $y_j \in H^2(e_j) \ \forall j \in J$ , and

$$\sum_{j\in J(v_i)} \partial y_j(v_i) = 0 \quad \forall i \in I.$$
(1)

Here  $\partial y_j(v_i)$  denotes the derivative of y at the vertex  $v_i$  taken along the edge  $e_j$  in the direction outwards the vertex. Vertex conditions (1) (together with continuity at v) are called the **standard** or **Kirchhoff-Neumann** (KN) conditions for the internal vertices. The continuity conditions are

$$y_j(v_i) - y_k(v_i) = 0 \quad \forall j, k \in J(v_i), \forall i \in I.$$
(2)

# Quantum graphs

Let q be a real valued function (potential) such that  $q|_{e_j} \in C[0, I_j]$ . We define the Schrödinger operator on the graph  $\Omega$  as the operator  $L = -\frac{d^2}{dv^2} + q$  in  $\mathcal{H}$  with the domain  $\mathcal{H}^2$ .

Changing q we change the operator L and, therefore, its spectrum and its multiplicity  $\sigma(\Omega, q)$ . The maximal possible multiplicity of an eigenvalue of L, denoted by  $\sigma(\Omega)$ , is very important for control and inverse problems on graphs.

The theory of PDEs on graphs started with pioneering papers by Lumer, von Below, Ali Mehmeti, and Nicaise in 1980-ies.

Control, observation and identification problems for DENs were studied almost exclusively on trees, see monographs (Lagnese, Leugering and Schmidt 1994; Avdonin and Ivanov 1995; Dáger and Zuazua 2006) and surveys (Avdonin 2008; Zuazua 2013). It is known that the IBVP for the wave equation on a tree graph is exactly controllable if the controls act at all or at all but one of the boundary vertices.



The wave equation on graphs with cycles is never exactly controllable from the boundary. It may be spectrally controllable, but this property is very unstable with respect to small perturbations of the system parameters.

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# Observation problem

We consider the following IBVP:

$$w_{tt} - w_{xx} + q(x)w = 0 \quad \text{in } \{\Omega \setminus V\} \times (0, T), \quad (3)$$

$$w_j(v_i, t) = w_k(v_i, t) \text{ for } j, k \in J(v_i), i \in I, t \in [0, T],$$
 (4)

 $\sum_{j\in J(v_i)} \partial w_j(v_i, t) = 0 \text{ at each vertex } i \in I, \ t \in [0, T],$  (5)

$$w|_{t=0} = w^0, \ w_t|_{t=0} = w^1 \quad \text{in } \ \Omega.$$
 (6)

Here T > 0,  $w^0 \in \mathcal{H}$ ,  $w^1 \in \mathcal{H}^{-1}$ . Note that (5) includes the boundary condition,  $\partial w|_{\Gamma} = 0$ . Using the Fourier method, one can show, similarly to (Avdonin and Nicaise 2015), that that for any  $i \in I, j \in J$ ,

$$w(v_i, \cdot) \in L^2(0, T), \ \partial w_j(v_i, \cdot) \in H^{-1}(0, T)$$

$$(7)$$

## Observation problem

We define a set of active vertices  $V^* = \{v_i : i \in I^*\}$  as a subset of V, where we put observers for the trace  $w(v, \cdot)$ . For each vertex  $v_i$  we define a set of active edges  $E^*(v_i)$  as a subset of  $E(v_i)$ , where we put observers for directional derivatives  $\partial w_j(v_i, \cdot), j \in J^*(v_i) \subset J(v_i)$ . Note that  $J^*(v_i)$  may be empty. Let  $E^* := \bigcup_{i \in I} E^*(v_i)$  and  $J^* := \bigcup_{i \in I} J^*(v_i)$ . We call  $\{V^*, E^*\}$  the **active set**. We say that the system (3)–(6) with the active set  $\{V^*, E^*\}$  is (exactly) **observable** in time T if there is a positive constant C, independent of  $w_0, w_1$ , such that

$$\sum_{i \in J^*} \|w(v_i, \cdot)\|_{L^2(0,T)}^2 + \sum_{j \in J^*} \|\partial w_j(v_i, \cdot)\|_{H^{-1}(0,T)}^2 \ge C\left\{\|w^0\|_{\mathcal{H}}^2 + \|w^1\|_{\mathcal{H}^{-1}}^2\right\}$$
(8)

for every  $w^0 \in \mathcal{H}, \ w^1 \in \mathcal{H}^{-1}.$ 

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# Control problem

$$u_{tt} - u_{xx} + q(x)u = 0 \quad \text{in} \quad \{\Omega \setminus V\} \times (0, T), \tag{9}$$

$$\sum_{j \in J(v_i)} \partial u_j(v_i, t) = \begin{cases} f_i^N(t), & i \in I^*, \\ 0, & i \in I \setminus I^*, \end{cases}$$

$$\begin{cases} u_j(v_i, t) - u_k(v_i, t) = f_j^D(t), & i \in I, \ j \in J^*(v_i), \ k \in J^c(v_i), \\ u_j(v_i, t) - u_k(v_i, t) = 0, & i \in I, \ j, k \in J^c(v_i), \end{cases}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in} \ \Omega. \tag{11}$$

We assume that  $f_i^N \in L^2(0, T), i \in I^*$ ;  $f_j^D \in H^1_*(0, T), j \in J^*$ . The space of controls  $L^2(0, T; \mathbb{R}^{|I^*|}) \times H^1_*(0, T; \mathbb{R}^{|J^*|}) =: \mathcal{F}^T$ . The system (9)–(12) is called *exactly controllable* in time *T* if  $\{(u^{\mathrm{f}}(\cdot, T), u_t^{\mathrm{f}}(\cdot, T)) : \mathrm{f} \in \mathcal{F}^T\} = \mathcal{H}^1 \times \mathcal{H}.$ 

# Multiplicity of the spectrum

How many controls we need for controllability?

$$u_t = Au + Bf, f(t) \in \mathbb{R}^m$$

$$c_k = \langle f, e_k \rangle_{L^2(0,T;\mathbb{R}^m)}, \quad e_k(t) = (B^* \varphi_k) e^{\lambda_k t}, \quad k = 1, \dots, n.$$

Number of controls should be greater than or equal to the multiplicity of the spectrum of *A*.

For distributed parameter systems it is a necessary condition of approximate controllability.

The maximal possible multiplicity of an eigenvalue of  $-d^2/dx^2 + q$  is denoted by  $\sigma(\Omega)$ . For trees  $\sigma(\Omega) = |\Gamma| - 1$ . (Kac and Pivovarchik 2011)

## The IBVP on a finite interval

$$\begin{cases} u_{tt} - u_{xx} + q(x)u = 0, & 0 < x < I, \ 0 < t < T\\ u|_{t \le 0} = 0, & u(0, t) = f(t), & u(I, t) = 0. \end{cases}$$
(13)  
When  $T \le I$ ,

$$u^{f}(x,t) = \begin{cases} 0, & 0 < t < x \\ f(t-x) + \int_{x}^{t} w(x,s)f(t-s) \, ds, & x \le t. \end{cases}$$
(14)

Here w(x, t) is a solution to the Goursat problem

$$\begin{cases} w_{tt} - w_{xx} + q(x)w = 0, & 0 < x < t \\ w(0, t) = 0, w(x, x) = -\frac{1}{2} \int_0^x q(s) \, ds \end{cases}$$
(15)

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### Shape, velocity and exact controllability

Shape controllability in T = l: for any function  $\phi$  such that  $\phi(l) = 0$ , there exists control function  $f \in H^1(0, l)$  such that  $u^f(x, l) = \phi(x)$ :

$$\phi(x) = f(l-x) + \int_{x}^{l} w(x,s)f(l-s) \, ds.$$
 (16)

Velocity controllability in T = l: for any function  $\psi$ , there exists a control g such that  $u_t^g(x, l) = \psi(x)$ :

$$\psi(x) = g'(l-x) + \int_{x}^{l} w(x,s)g'(t-s) \, ds.$$
 (17)

Exact controllability in T = 2I: for any  $\phi$  and  $\psi$ , there exists h such that  $u^h(x, 2I) = \phi(x)$  and  $u^h_t(x, T) = \psi(x)$ .

Shape/velocity controllability  $\Leftrightarrow$  solvability of the moment problems

$$a_n = \int_0^T f(t) \sin \omega_n (T-t) dt, \quad \forall \{a_n\} \in \ell^2,$$
  
 $b_n = \int_0^T g(t) \cos \omega_n (T-t) dt, \quad \forall \{b_n\} \in \ell^2.$ 

We extend f and g to [0,2T]: f(2T-t) = -f(t), g(2T-t) = g(t), and put  $h(t) = \frac{1}{2}[f(t)+g(t)]$ . Then

$$a_n = \int_0^{2T} h(t) \sin \omega_n (T-t) dt, \quad b_n = \int_0^{2T} h(t) \cos \omega_n (T-t) dt.$$

$$\tilde{a}_n = \int_0^{2T} h(t) \sin \omega_n (2T - t) \, dt, \quad \tilde{b}_n = \int_0^{2T} h(t) \, \cos \omega_n (2T - t) \, dt.$$

### Controllability on trees

 $\Gamma_1$  contains all or all but one boundary vertices.

Let U be a union of disjoint paths (except for the end points) from a controlled vertex to a point in a finite tree graph  $\Omega$  such that  $\bigcup_{P \in U} P = \Omega$ . We put

 $T_0 = \min_{U} \max_{P \in U} \text{ length } P.$ 

#### Theorem

For any  $T \ge 2T_0$  and any  $y \in \mathcal{H}$ ,  $z \in \mathcal{H}^{-1}$ , there exists  $f \in \mathcal{F}^T$ such that  $u^f(\cdot, T) = y$  and  $u^f_t(\cdot, T) = z$ . For any  $T \ge T_0$  we get either  $u^f(\cdot, T) = y$  (shape controllability) or  $u^f_t(\cdot, T) = z$  (velocity controllability).

(Avdonin and Zhao 2019)

## Union path representation of a tree





Controls act on all boundary vertices

Controls act on all but one boundary vertices

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# A control problem on a lasso graph

$$u_{tt} - u_{xx} = 0,$$
  

$$u(\cdot, 0) = u_t(\cdot, 0) = 0,$$
  

$$\partial u_1(l, t) = f(t),$$
  

$$u_2(0, t) - u_1(0, t) = h(t), \ u_3(0, t) - u_1(0, t) = 0, \ \sum_{j=1}^3 \partial u_j(0, t) = 0.$$
  

$$x = l \qquad e_1 \qquad e_2 \qquad x = a$$

$$\mathcal{H} := L^{2}(\Omega). \ \mathcal{H}^{1} := \{ y : y |_{e_{j}} \in H^{1}(e_{j}), y_{1}(0) = y_{2}(0) = y_{3}(0) \}.$$

# Acyclic orientation

Start with an acyclic orientation



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#### Control Theory for DENs

# Placing Dirichlet and Neumann controls on $\vec{\Omega}$



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# Path union

Let  $\vec{\Omega}$  be a DAG of  $\Omega$ . Let *U* be a union of directed paths satisfiing the conditions:

The direction of all edges are the same as the direction of all paths they are on.

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- 2 All paths are disjoint except for the starting and finishing vertices.
- 3 Each path begins with an active vertex or active edge.
  4 Ω = ∪<sub>P∈U</sub>P.

### Minimal number of controllers

The sharp estimate of the minimal number of controllers  $\varkappa(\Omega)$  that guarantees the exact controllability of the wave equation on the graph for all q.

A **cut-vertex** is a vertex whose deletion increases the number of connected components of a graph. Deletion of all cut-vertices separates  $\Omega$  into blocks. The blocks sharing only one cut-vertex with its complement are called **pendant blocks**. We denote the number of pendant blocks by  $\beta$ .

$$\varkappa(\Omega) = \mu + 1$$
, if  $\beta = 1$ ;  $\varkappa(\Omega) = \mu + \beta - 1$ , if  $\beta \ge 2$ . (18)

Here  $\mu = |E| - |V| + 1$  is the cyclomatic number of  $\Omega$ .

# Comparing $\varkappa(\Omega)$ with $\sigma(\Omega)$

Let  $p^t$  be the numbers of boundary vertices for the tree obtained by contracting all the cycles of the graph into vertices

$$\begin{aligned} \sigma(\Omega) &= \mu + 1, \text{ if } \Omega \text{ is cyclically connected;} \\ \sigma(\Omega) &= \mu + p^t - 1, \text{ if } \Omega \text{ is not cyclically connected.} \end{aligned}$$

Since 
$$\beta \geq p^t$$
,  $\varkappa(\Omega) \geq \sigma(\Omega)$ .

 $\varkappa(\Omega) = \sigma(\Omega)$  for trees, for a ring, the lasso graph, and in many other cases.

The talk is based in part on the joint work with Y. Zhao.

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