Logarithmic convexity and impulsive controllability for the heat equation with dynamic boundary conditions

Bordeaux

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Abstract

In this poster, we prove a logarithmic convexity that reflects an observability estimate at a single point of time for 1-D heat equation with dynamic boundary conditions. Consequently, we establish the impulse approximate controllability for the impulsive heat equation with dynamic boundary conditions. Moreover, we obtain an explicit upper bound of the cost of impulse control. At the end, we present some numerical tests to validate the theoretical results.

1 Introduction

It is well-known that the heat equation is one of the most significant partial differential equations of parabolic type, it is a model for a large class of physical phenomena which describes the distribution of heat in a given region over time. In general a linear heat equation with a control function can be presented as follows:

$$\begin{cases} \partial_t \psi - \Delta\psi = u(x,t)\mathbf{1}_\omega, & \text{in } \Omega\times(0,T) \\ B.C \\ I.C \end{cases} \tag{1}$$

♣ u is the control acting on ω of Ω.

The theory of impulsive differential equations was initiated by **V. D. Mil'man and A. Mishkis** in 1960 [2]. Afterward, many scientists contributed to the enrichment of this theory. For instance **K.D. Phung** [3] who prove the approximately controllability for the following impulsive heat equation

$$\begin{cases} \partial_t \psi - \Delta\psi = 0, & \text{in } \Omega\times(0,T)\backslash\{\tau\} \\ \psi = 0, & \text{on } \partial\Omega\times(0,T) \\ \psi(\cdot,0) = \psi^0, & \text{in } \Omega \\ \psi(\cdot,\tau) = \psi\left(\cdot,\tau^-\right) + 1_\omega h, & \text{in } \Omega \end{cases} \tag{2}$$

Also, We mention A. Khapalov [1] who proved the exact controllability of a class of second-order hyperbolic boundary problems with impulse controls using the Huygens' principle.

To better understand this phenomenon of impulses we have made some simulations for the 1-D heat equation with Dirichlet boundary conditions

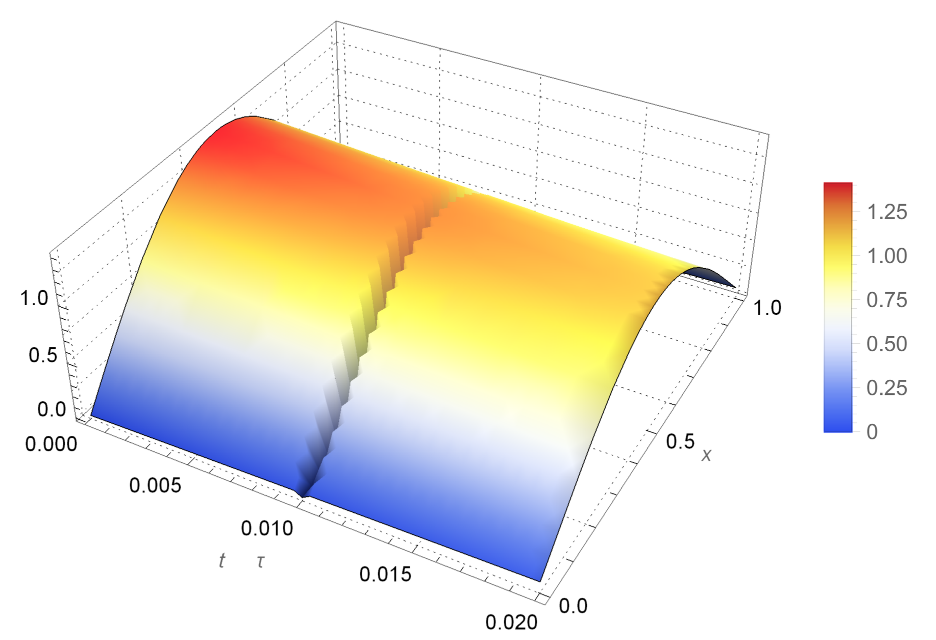

Figure 1: The instant of pulse is $\tau = 0.01$

Motivated by the above works, our interest is to investigate the impulse controlled heat equation with dynamic boundary conditions given by

$$\begin{cases} \partial_t\psi(x,t) - \partial_{xx}\psi(x,t) = 0, & \text{in } (a,b)\times(0,T)\backslash\{\tau\}, \\ \psi(x,\tau) = \psi\left(x,\tau^-\right) + \mathbb{1}_\omega(x)h(x), & \text{in } (a,b), \\ \partial_t\psi(a,t) - \partial_x\psi(a,t) = 0, & \text{on } (0,T)\backslash\{\tau\}, \\ \partial_t\psi(b,t) + \partial_x\psi(b,t) = 0, & \text{on } (0,T)\backslash\{\tau\}, \\ \psi(a,\tau) = \psi\left(a,\tau^-\right), & \\ \psi(b,\tau) = \psi\left(b,\tau^-\right), & \\ (\psi(x,0),\psi(a,0),\psi(b,0)) = \left(\psi^0(x),c,d\right), & \text{in } (a,b), \end{cases} \tag{3}$$

where $(a,b)\subset\mathbb{R}$ is an open interval, $T>0$ is the final time, $\tau\in(0,T)$ is an impulse time, $\left(\psi^0,c,d\right)\in L^2(a,b)\times\mathbb{R}^2$ denotes the initial data, $\psi(\cdot,\tau^-)$ denotes the left limit of the function ψ at time τ, and $\omega\Subset(a,b)$ is a nonempty open subset.

2 Well-posedness

In what follows, we will often use the following real Hilbert space $\mathbb{L}^2 := L^2(a,b)\times\mathbb{R}^2$, equipped with the inner product

$$\langle (u,c,d),(v,c_1,d_1)\rangle = \langle u,v\rangle_{L^2(a,b)} + cc_1 + dd_1.$$

The previous system can be presented as the following impulsive Cauchy problem

$$\text{(ACP)} \quad \begin{cases} \partial_t\Psi(t) = \mathbf{A}\Psi(t), & (0,T)\setminus\{\tau\}, \\ \triangle\Psi(\tau) = (\mathbb{1}_\omega h,0,0), \\ \Psi(0) = (\psi^0,c,d), \end{cases}$$

where $\Psi := (\psi,\psi(a,\cdot),\psi(b,\cdot))$ and $\triangle\Psi(\tau) := \Psi(\cdot,\tau) - \Psi\left(\cdot,\tau^-\right)$. For all $\Psi_0 := (\psi^0,c,d)\in\mathbb{L}^2$, the system (ACP) has a unique mild solution given by

$$\Psi(t) = \mathrm{e}^{t\mathbf{A}}\Psi_0 + \mathbb{1}_{\{t\geq\tau\}}(t)\,\mathrm{e}^{(t-\tau)\mathbf{A}}(\mathbb{1}_\omega h,0,0), \qquad t\in(0,T).$$

3 Observability at one point of time

The key lemma that will enable us to prove the impulsive approximate controllability of the above equation is the following observability estimate at one point of time.

Lemma 3.1. *Let $\omega\Subset(a,b)$ be an open nonempty set. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product of $L^2(a,b)\times\mathbb{R}^2$ and $\|\cdot\|$ be its corresponding norm. Then the following logarithmic convexity estimate holds*

$$\|U(\cdot,T)\| \leq \left(\mu\mathrm{e}^{\frac{K}{T}}\|u(\cdot,T)\|_{L^2(\omega)}\right)^\beta \|U(\cdot,0)\|^{1-\beta}, \tag{4}$$

where $\mu, K>0$, $\beta\in(0,1)$ and $U=(u(\cdot,\cdot),u(a,\cdot),u(b,\cdot))$ is the solution of the following system

$$\begin{cases} \partial_t u(x,t) - \partial_{xx}u(x,t) = 0, & \textit{in } (a,b)\times(0,T), \\ \partial_t u(a,t) - \partial_x u(a,t) = 0, & \textit{on } (0,T), \\ \partial_t u(b,t) + \partial_x u(b,t) = 0, & \textit{on } (0,T), \\ (u(x,0),u(a,0),u(b,0)) = \left(u^0(x),c_1,d_1\right), & \textit{in } (a,b). \end{cases} \tag{5}$$

Remark 3.1. *We emphasize that* (4) *is an observability inequality estimating the whole solution $U=(u(\cdot,\cdot),u(a,\cdot),u(b,\cdot))$ of system* (5) *at terminal time T by only using one internal observation on the first component u, which is localized in the subset ω.*

4 Approximate impulse controllability

Control theory deals with how an arbitrary initial state can be directed exactly or approximately close to a given final state using a set of admissible controls. In this section, we study the impulse approximate controllability for the system below. Here, the control function acts on a subdomain ω and at one point of time $\tau\in(0,T)$. Next, we state the main result on approximate impulse controllability for system (3).

Theorem 4.1. *The system* (3) *is approximate impulse controllable at any time $T>0$. Moreover, for any $\varepsilon>0$, the cost of approximate impulse control satisfies*

$$K(T,\varepsilon) \leq \frac{\mathcal{M}_1\mathrm{e}^{\frac{\mathcal{M}_2}{T-\tau}}}{\varepsilon^\delta},$$

where the positive constants $\mathcal{M}_1$, $\mathcal{M}_2$ and δ are positive constants.

5 Numerical experiments

Next, we present some numerical tests to illustrate our theoretical results. This will be done by computing the HUM impulse controls based on a penalized HUM approach combined with a CG algorithm.

In the numerical tests, we will choose the following values

$$T=0.02, \quad \tau=0.01, \quad a=0,\ b=1, \quad \omega=(0.2,0.8)\Subset(0,1),$$

and we consider the initial datum to be controlled as

$$\psi_0(x) = \sqrt{2}\sin(\pi x), \qquad x\in[0,1].$$

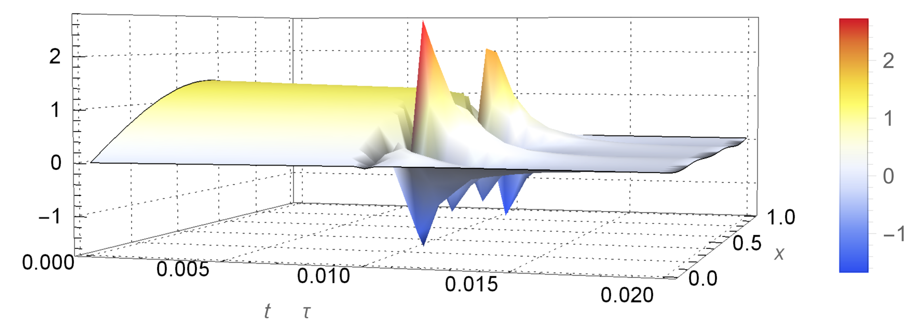

Figure 2: The controlled solution.

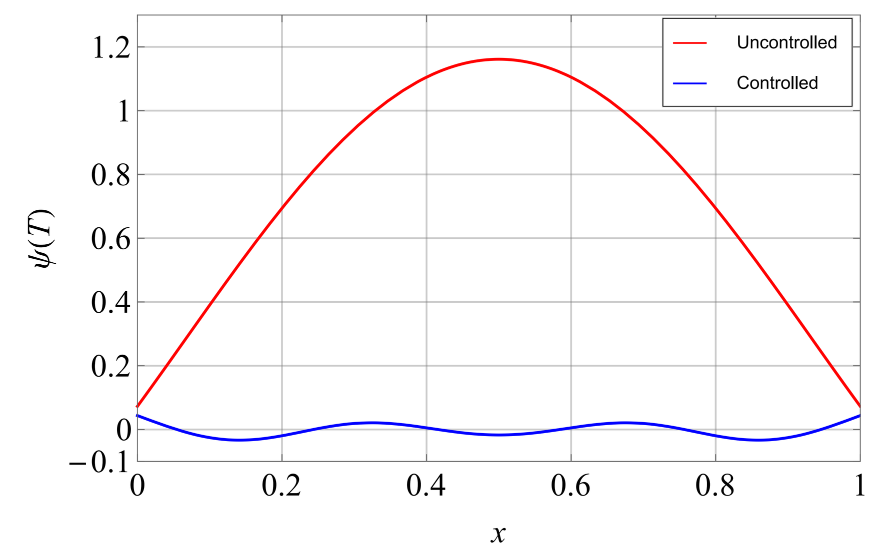

Figure 3: The final state for uncontrolled and controlled solutions.

ε	10^{-2}	10^{-3}	10^{-4}
N_{iter}	6	10	25
$\|\Psi(T)\|$	8.54×10^{-2}	7.27×10^{-2}	6.47×10^{-2}
$\|h\|_{L^2(\omega)}$	0.9478	1.1325	2.2109

Figure 4: Numerical results for $T=0.02$, $\tau=0.01$ and $tol=10^{-3}$.

We clearly see that the distance $\|\Psi(T)\|$ to the target zero decreases and the norm of the impulse control $\|h\|_{L^2(\omega)}$ increases as ε diminishes.

Conclusions

In this work, a logarithmic convexity result has been proved for the 1-D heat equation with dynamic boundary conditions. As an application, the impulsive approximate controllability for the system (3) has been established with an explicit bound of the cost. The proof is based on the Carleman commutator approach. Afterward, a constructive algorithm has been developed to numerically construct the impulse control of minimal L^2-norm. This has been done by combining a penalized HUM approach and a CG method. Finally, a numerical simulation has been performed to illustrate the theoretical result of impulse approximate controllability.

To the best of the authors knowledge, dynamical systems with impulsive controls have not been much studied numerically, which opens the doors to many possibilities for dealing with such problems. This work can be generalized in several ways, for instance, one would study the case of an infinite number of impulses or the case of some perturbations as: nonlinearities, delays and non local conditions. One would also change the type of impulses by considering non-instantaneous impulses. Such problems would be of much interest to investigate.

References

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