Nonlinear shallow water model for an oscillating water column with time-dependent air pressure

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Introduction

In this work we present a mathematical model and simulations of a particular wave energy converter, the so-called oscillating water column (OWC). In this device, waves governed by the one-dimensional nonlinear shallow water equations arrive from offshore, encounter a step in the bottom and then arrive into a chamber to change the volume of the air to activate a turbine, which produce electrical energy. The system is reformulated as two transmission problems: one is related to the wave motion over the stepped topography and the other one is related to the wave-structure interaction at the entrance of the chamber. By using Riemann invariants, and the Lax-Friedrichs scheme we also get numerical simulations in a simplified case. A graphical sketch of the configuration of the OWC device is presented in the following figure

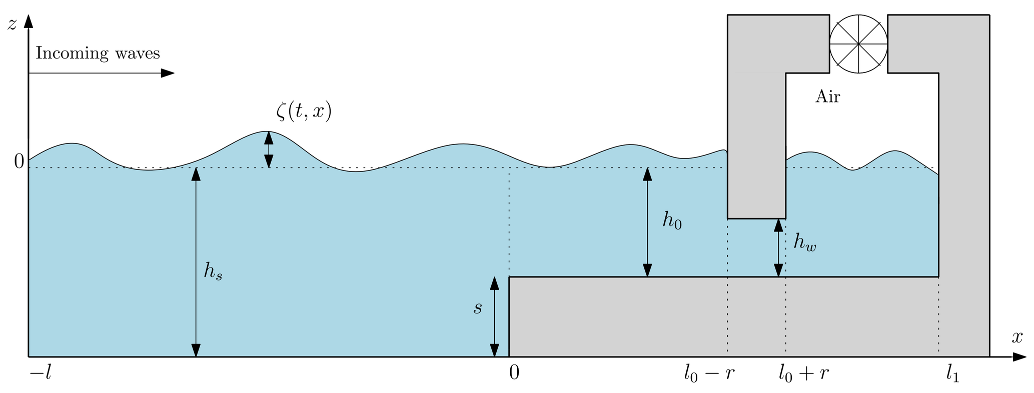

Figure: Configuration

We consider an incompressible, irrotational, inviscid and homogeneous fluid in a shallow water regime, which occurs in the region where the OWC is installed. The motion of the fluid is governed by the 1D nonlinear shallow water equations (NSW)

$$\begin{cases}\partial_t\zeta+\partial_x q=0\\ \partial_t q+\partial_x\left(\dfrac{q^2}{h}\right)+gh\partial_x\zeta=-\dfrac{h}{\rho}\partial_x\underline{P}\end{cases}\qquad(1)$$

for $x\in(-l,l_1)$, where $\zeta(t,x)$ is free surface elevation, $h(t,x)$ is the fluid height, ρ is the fluid density, $\underline{P}$ is the surface pressure of the fluid and $q(t,x)$ is the horizontal discharge defined by $q(t,x)=\int_{-h_0}^{\zeta(t,x)}u(t,x,z)dz$, where $u(t,x,z)$ is the horizontal component of the fluid velocity vector field. The non-constant pressure inside the OWC chamber is modelled by considering a perturbation function $P_{ch}(t)$, which satisfies the ordinary differential equation:

$$\frac{d}{dt}P_{ch}+\frac{\gamma P_{atm}}{h_{ch}K}P_{ch}=\frac{\gamma P_{atm}}{h_{ch}}\frac{d}{dt}\overline{\zeta},\quad P_{ch}(0)=P_{ch,0},$$

where $\overline{\zeta}(t)$ is the spatially averaged surface elevation. The system is completed by considering an initial configuration where the fluid is at rest.

Governing equations

The model proposed here is presented as two transmission problems; the first one governing the dynamic between the wave motion over a discontinuous topography with the wave-structure interaction at the entrance of the chamber, and the second one considering the wave motion in the chamber.
These partial differential equations systems read as:

1. In $(-l,0)$, $h=h_s+\zeta$, $\underline{P}=P_{atm}$ and
$$\begin{cases}\partial_t\zeta+\partial_x q=0\\ \partial_t q+\partial_x\left(\frac{q^2}{h}\right)+gh\partial_x\zeta=-\frac{h}{\rho}\partial_x P_{air}=0\end{cases}$$

First transmission problem:

$$\updownarrow\ \zeta_{x=0^-}=\zeta_{x=0^+},\quad q_{x=0^-}=q_{x=0^+}$$

2. In $(0,l_0-r)$, $h=h_0+\zeta$, $\underline{P}=P_{atm}$ and
$$\begin{cases}\partial_t\zeta+\partial_x q=0\\ \partial_t q+\partial_x\left(\frac{q^2}{h}\right)+gh\partial_x\zeta=-\frac{h}{\rho}\partial_x P_{air}=0\end{cases}$$

Second transmission problem:

$$\updownarrow\ [\![q]\!]=0,\quad P_{ch}(t)+\left[\!\!\left[\frac{\rho}{2}\frac{q^2}{h^2}+\rho g\zeta\right]\!\!\right]=-\rho\frac{2r}{h_w}\frac{d}{dt}q_{\mathcal{I}}$$

3. In (l_0+r,l_1), $h=h_0+\zeta$, $\underline{P}=P_{atm}+P_{ch}(t)$ and
$$\begin{cases}\partial_t\zeta+\partial_x q=0\\ \partial_t q+\partial_x\left(\frac{q^2}{h}\right)+gh\partial_x\zeta=-\frac{h}{\rho}\partial_x P_{air}=0\end{cases}$$

where for an arbitrary function f, we have
$$[\![f]\!]=f_{|x=l_0+r}-f_{|x=l_0-r}.$$
The local wellposedness of the transmission problems above is studied in [2].

Discretizations

By considering $P_{ch}(t)=0$, in the following we obtain a numerical solution of the transmission problems described above. To this end, we rewrite the nonlinear shallow water equations (1) in a compact form by introducing the couple $U=(\zeta_e,q_e)^T$:

$$\partial_t U+A(U)\partial_x U=0,\qquad(2)$$

where $A(U)=\begin{pmatrix}0 & 1\\ gh_e-\frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e}\end{pmatrix}$.

We remark that, the associated eigenvectors $e_+(U)$ and $e_-(U)$ of $A(U)$ are given by

$$e_+(U)=\Big(\sqrt{gh_e}-\frac{q_e}{h_e},1\Big)^T,\quad e_-(U)=\Big(-\sqrt{gh_e}-\frac{q_e}{h_e},1\Big)^T.$$

Hence, by taking the scalar product of (2) and $e_\pm(U)$ we obtain

$$\partial_t\Big(2\sqrt{gh_e}\pm\frac{q_e}{h_e}\Big)\pm\Big(\sqrt{gh_e}\pm\frac{q_e}{h_e}\Big)\Big[\partial_x\Big(2\sqrt{gh_e}\pm\frac{q_e}{h_e}\Big)\Big]=0$$

and then, we can recast the 1D NSW equations as the two following transport equations on R and L:

$$\partial_t R+\lambda_+(U)\partial_x R=0,\qquad \partial_t L-\lambda_-(U)\partial_x L=0.$$

The finite volume method is a standard discretization approach for partial differential equations, especially those that arise from conservation laws. We first rewrite equation (2) as the following conservative form

$$\partial_t U+\partial_x(F(U))=0\qquad(3)$$

with

$$F(U)=\Big(q_e,\frac{1}{2}g(h_e^2-h_0^2)+\frac{q_e^2}{h_e}\Big)^T.$$

By means of a finite volume approach, the equation can be discretized as

$$\frac{U_i^{m+1}-U_i^m}{\delta_t}+\frac{\left(F^m_{i+1/2}-F^m_{i-1/2}\right)}{\delta_x}=0$$

where the flux F is discretized with cell centres indexed as i and cell edge fluxes indexed as $i\pm1/2$. The choice of $F^m_{i\pm1/2}$ depends on the numerical scheme. We consider here the well-known Lax–Friedrichs scheme to get the discrete flux, with $F^m_i=F(U^m_i)$,

$$F^m_{i-1/2}=\frac{1}{2}\left(F^m_i+F^m_{i-1}\right)-\frac{\delta_x}{2\delta_t}\left(U^m_i-U^m_{i-1}\right).$$

Simulations

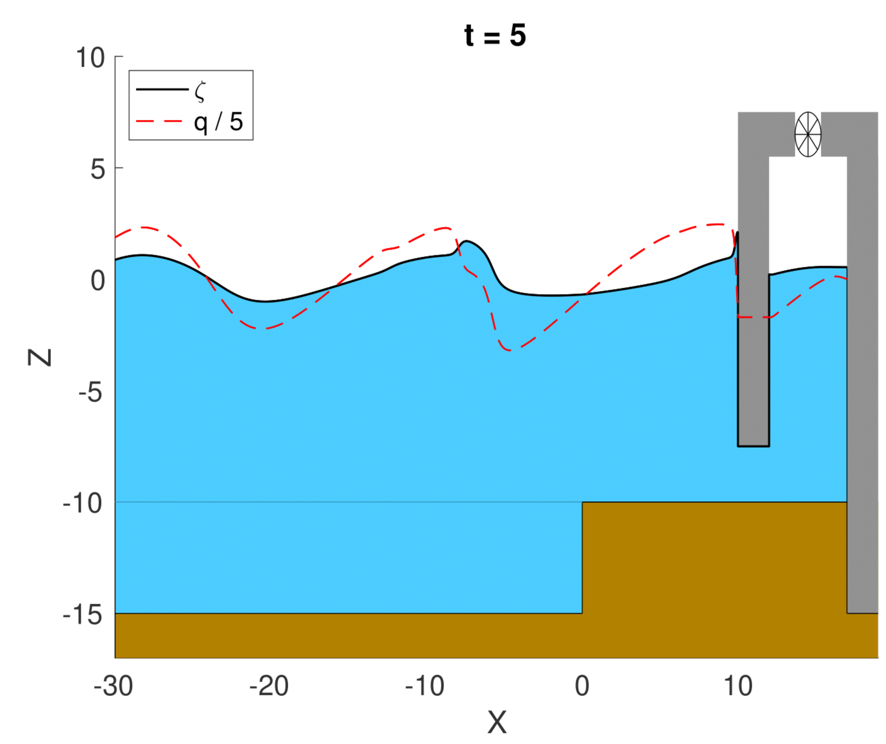

Figure 2: Numerical simulation of the transmission problems at time $t=5\,$s.

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