Boundary control problem of a water waves system in a tank

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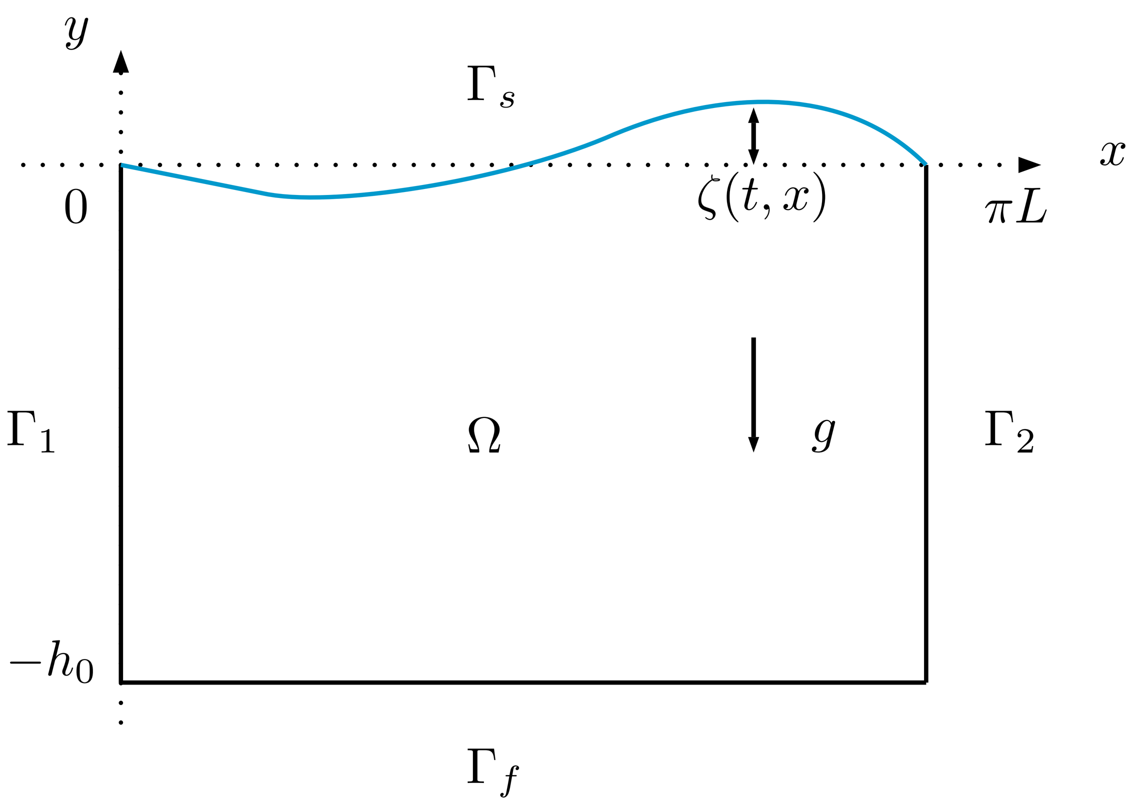
Problem setting

Consider the small amplitude gravity water waves in a 2-D rectangular domain Ω. The control u on a lateral boundary is used to regulate the acceleration field of the fluid imposed by a rigid wave maker.
Question: How to describe the decay rate of the elevation of the free surface in a pool? How does it behaves when taking the shallowness limit?



$\zeta(t,x)$: The elevation of the free surface;
$h(y)$: The profile of the active boundary, say Γ_1;
Rigid wave maker: The horizontal velocity $v = h(y)w(t)$ and $u = \dot{w}(t)$,
Shallow water regime: the shallowness parameter $\mu = \frac{h_0^2}{L^2} \ll 1$.

Basic concepts

1. **A well-posed linear control system** (A, B) **on** X:
- A is a generator of a C_0-semigroup $(\mathbb{T}_t)_{t\geq 0}$ on X;
- B is an admissible control operator for $\mathbb{T}$, i.e.

$$\operatorname{Ran}\Phi_\tau \subset X, \quad \text{with} \quad \Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-s}Bu(s)ds, \quad \forall \tau \geq 0.$$

2. **Stabilizability of the system** (A, B) **with bounded feedback** $F \in \mathcal{L}(X, U)$: Let $\mathbb{T}^{cl}$ be the C_0-semigroup generated by $A + BF$.
- Exponentially stabilizable:

$$\|\mathbb{T}^{cl}z\|_X \leq Ce^{-\alpha t}\|z\|_X, \quad (\alpha > 0),$$

- Strongly stabilizable:

$$\lim_{t\to 0}\|\mathbb{T}^{cl}z\| = 0, \quad \forall z \in X,$$

- Uniformly stabilizable for smooth data (USSD):

$$\|\mathbb{T}^{cl}z\|_X \leq f(t)\|z\|_{\mathcal{D}(A)}, \quad \forall z \in \mathcal{D}(A).$$

where $f : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} f(t) = 0$.

Main results [3][4]

1. Let $h \in L_0^2[-1,0]$. For every $u \in L_{loc}^2[0,+\infty)$, there exists a well-posed linear control system (A, B) with the state $z \in X = H_{\frac{1}{2}} \times H$ and input $u \in U = \mathbb{C}$, s.t. z solves

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ z(0) = z_0 \end{cases} \tag{8}$$

with $z(t) = \begin{bmatrix} \zeta(t,\cdot) \\ \dot{\zeta}(t,\cdot) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}.$

2. The well-posed linear control system $(\mathbb{T}, \Phi)$ is strongly stabilizable, but it fails the exponential stabilizability; $(\mathbb{T}, \Phi)$ is USSD if

$$\inf_{k\in\mathbb{N}} \frac{k}{\cosh k}\left|\int_{-1}^0 h(y)\cosh[k(y+1)]\mathrm{d}y\right| > 0$$

s.t. $\mathbb{T}^{cl}$ (with generator $A - BB^*$) satisfies

$$\|\mathbb{T}_t^{cl}z_0\|_X \leq (1+t)^{-\frac{1}{6}}\|z_0\|_{\mathcal{D}(A)}, \quad \forall z \in \mathcal{D}(A). \tag{9}$$

3. For any initial data $\zeta_0 \in \mathcal{H}^1[0,\pi]$ and $\zeta_1 \in L^2[0,\pi]$, let ζ_μ be the solution of the solution of the water waves and let ζ be the solution of the wave equation with Neumann boundary control. Then we have

$$\lim_{\mu\to 0}\sup_{t\in[0,\tau]}\|\zeta_\mu - \zeta\|_{\mathcal{H}^{\frac{1}{2}}[0,\pi]} = 0,$$

$$\lim_{\mu\to 0}\sup_{t\in[0,\tau]}\|\partial_t\zeta_\mu - \partial_t\zeta\|_{L^2[0,\pi]} = 0.$$

Governing equations

Let $\phi(t,x,y)$ be the velocity potential (for irrotational field). The motion of a inviscid, imcompressible and irrotational fluid is described by the free surface Bernoulli equations [2]:

$$\Delta\phi = 0 \quad \text{in} \quad \Omega, \tag{1}$$

and (P_{atm}: the (constant) atmospheric pressure)

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + gy = -\frac{P - P_{atm}}{\rho} \quad \text{in} \quad \Omega \tag{2}$$

The continuity of the pressure across the interface (without surface tension):

$$P = P_{atm} \quad \text{on} \quad \Gamma_s. \tag{3}$$

Based on the small-amplitude assumption (drop nonlinear terms) we obtain from (2) that

$$\partial_t\phi + g\zeta = 0 \quad \text{on} \quad \Gamma_s, \tag{4}$$

Boundary equations ($\vec{n}$: the outer normal):

$$\begin{aligned} &\text{(kinematic condition)} && \partial_{\vec{n}}\phi = \partial_t\zeta \quad \text{with} \quad \vec{n} = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix} \quad \text{on } \Gamma_s, \\ &\text{(imposed wave maker)} && \partial_{\vec{n}}\phi = v(t,y) \quad \text{on} \quad \Gamma_1, \\ &\text{(impermeable condition)} && \partial_{\vec{n}}\phi = 0 \quad \text{on} \quad \Gamma_2 \cup \Gamma_f. \end{aligned} \tag{5}$$

- Difficulties: non-smooth domain, mixed boundary condition

Two important operators

1. **Partial Dirichlet and Partial Neumann map** [1]:
Assume that Φ and Ψ satisfy:

$$\begin{cases} \Delta\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \varphi & \text{on} \quad \Gamma_s, \\ \partial_{\vec{n}}\Phi = 0 & \text{on} \quad \Gamma_1 \cup \Gamma_2 \cup \Gamma_f. \end{cases}, \quad \begin{cases} \Delta\Psi = 0 & \text{in} \quad \Omega, \\ \Psi = 0 & \text{on} \quad \Gamma_s, \\ \partial_{\vec{n}}\Psi = \psi & \text{on} \quad \Gamma_1, \\ \partial_{\vec{n}}\Psi = 0 & \text{on} \quad \Gamma_2 \cup \Gamma_f. \end{cases} \tag{6}$$

Partial Dirchlet map $D : \varphi \longmapsto \Phi$, Partial Neumann map $N : \psi \longmapsto \Psi$.
2. **Dirichlet to Neumann** A_0 **and Neumann to Neumann** B_0:

$$A_0 : \varphi \longmapsto \partial_{\vec{n}}\Phi\big|_{\Gamma_s}, \qquad B_0 : \psi \longmapsto \partial_{\vec{n}}\Psi\big|_{\Gamma_s}. \tag{7}$$

3. **Properties**:
Let $H = L_0^2(0,\pi) = \left\{f \in L^2(0,\pi) \,\middle|\, \int_0^\pi f(x)\mathrm{d}x = 0\right\}$ and let $\mathcal{H}^1(\Omega)$ be the classical Sobolev spaces. $\left\{\varphi_k(x) = \sqrt{\frac{2}{\pi}}\cos(kx)\right\}_{k\in\mathbb{N}}$ forms an orthonormal basis in H. Define a series of Hilbert spaces by

$$H_\alpha = \left\{\eta \in H \,\middle|\, \sum_{k\in\mathbb{N}} k^{2\alpha}|\langle\eta, \varphi_k\rangle|^2 < \infty\right\} \qquad (\alpha \geq 0).$$

- The partial Dirichlet map $D \in \mathcal{L}(L^2[0,\pi], L^2(\Omega))$;

The partial Neumann map $N \in \mathcal{L}(L^2[-1,0], \mathcal{H}_{top}^1(\Omega))$ with

$$\mathcal{H}_{top}^1(\Omega) = \left\{f \in \mathcal{H}^1(\Omega) \,\middle|\, f(x,0) = 0, x \in (0,\pi)\right\};$$

- $A_0 : H_1 \to H$ is strictly positive and for each $k \in \mathbb{N}$

$$A_0\varphi_k = \lambda_k\varphi_k, \qquad \lambda_k = k\tanh(k) \quad \forall\, k \in \mathbb{N}.$$

$B_0 \in \mathcal{L}(L^2[-1,0], L^2[0,\pi])$. In our case, $B_0 \in \mathcal{L}(\mathbb{C}, H)$ since $\psi = h(y)u(t)$ and $h \in L_0^2[-1,0]$ which ensures the conservation of volume.

Perspectives

- The similar problem on general convex domain;
- Nonlinear shallow water waves in a canal with a boundary control;
- The interaction between surface water waves and a floating body in the shallow water regime.

References

[1] Mottelet, S. (2000). Controllability and stabilization of a canal with wave generators. *SIAM J. on Control and Optim.*, 38(3), 711–735.
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