Introduction

Consider for \( t \geq 0 \) the Linear Time Invariant (LTI) system
\[
\dot{x}(t) = Ax(t) + Bu(t).
\]

- State controllability

\[ \forall x_0 \quad \exists u \quad \forall x_1 \quad x_0(x_0, t) \]  

Topic very well understood from literature.

Particularity

All the state variables are controlled.

Motivation

- What if you do not will to control all the state variables?

\[
\begin{array}{c}
\xrightarrow{f_2} \quad f_1 \quad \xrightarrow{u_2} \quad u_1 \\
\xrightarrow{q_2} \quad y_2 \quad \xrightarrow{q_1} \quad y_1
\end{array}
\]

- More generally

\[
\text{Input} \quad u \quad \xrightarrow{\text{System}} \quad y
\]

Questions:

- Can we reach any benchmark output \( y_{\text{ref}} \)?
- If YES, what is the suitable \( u \) to be used?

To answer these questions, Bertram and Sarachik introduced in [1] the notion of Output controllability.

Framework: LTI systems

We assume that the output is given by
\[
y(t) = Cx(t) + Du(t),
\]
\( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^q \).

Discussed notions:

State to output controllability

State to output controllability can be found in [2]. In these notions, we discuss how to steer an initial output value to a prescribed output.

Theorem (Criteria)

The system (1) is state to output controllable if and only if one of the following conditions is fulfilled

(i) There exists a time \( T > 0 \) such that the linear map \( E^T_y : \mathcal{C}([0, T], \mathbb{R}^m) \to \mathbb{R}^q \), defined by
\[
E^T_y(u) = \int_0^T C e^{(T-t)A} Bu(t) dt + Du(T)
\]
is surjective.

(ii) \( rk \left[ DB \quad CAB \cdots \quad CA^{m-1}B \right] = q \). (Given in [3])

(iii) \( K_T := C \int_0^T e^{(T-t)A} B e^{At} dt C^T + DD^T > 0 \) for some \( T > 0 \). (Also given in [3])

(iv) \( rk(C|D) = q \) and \( \lim \left( \frac{C}{D} \right)^T \in \left( \bigcup_{q_{k-1} = 1}^{\infty} \ker B^i(A^T)^j \right) \) \( A_T = A - \lambda I_n \) and \( m_A \), the algebraic multiplicity of \( \lambda \) in the minimal polynomial of \( A \).

(v) \( rk(C|D) = q \) and \( \left( \begin{array}{c} K_{A_T} \cdots 0 \\ 0 K_{A_T} \cdots 0 \\ \vdots \vdots \\ 0 \cdots 0 K_{A_T} \end{array} \right) (C|D)^j = (n + m) \rho \), where \( \lambda_1, \lambda_2, \ldots, \lambda_p = \sigma(A) \).

(vi) \( G_T^y := \int_0^T H_d(T, t) H_o(T, t) dt > 0 \) for some \( T > 0 \), where \( H_d(T, t) = C \int_0^T e^{(T-t)A} B e^{At} dt + D \).

Application to the cars example.

Consider for \( t \geq 0 \), system (1) with \( f_1 = -\alpha y_2 \) where \( y_2 \) is the speed of car \( l \) for \( l = 1, 2 \) and \( m_2 \) stands for the mass of car \( l \). Choosing \( m_1 = m_2 = 1 \) and \( \alpha_1 = \alpha_2 = 1/2 \) the system is given by
\[
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

- One can see that this system is state to output controllable in any time \( T > 0 \).
- Goal: steer the system from \( x_0 = (1, 0, 1, 0)^T \) to \( y_1 = (2, 0, 0, 1)^T \) in time \( T = 1 \).

Applying (2) with \( u_0 = (1, 0, 0)^T \), we get
\[
u = (u_1, u_2)^T, \quad u_1 = 1 - u_2, \quad u_2(t) = at + bt^2 + c(t^3 - 1) \quad \text{where} \quad a = (254 + 98e - 316\epsilon_1)/d, \quad b = (105 + 45e - 138\epsilon_1)/d, \quad \text{and} \quad c = (54 - 32\epsilon_1)/d, \quad \text{with} \quad d = 222e - 736\epsilon_1 + 610.
\]

Using the matrix \( K_T \), we get \( u = (u_1, u_2)^T, \quad u_1 = -u_2 \) and \( u_2(t) = (2\epsilon_1^2 - \epsilon_2^3 - 1)/(6\epsilon_1^2 - 10) \) for \( t \in [0, 1] \) and \( u(1) = 0 \).

References