Output controllability for Linear Time Invariant systems

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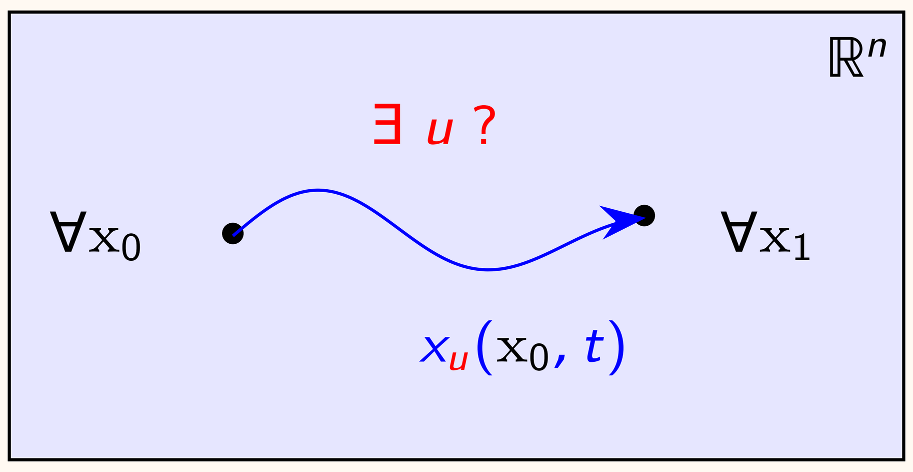
Introduction

Consider for $t \geq 0$ the Linear Time Invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t).$$

- State controllability



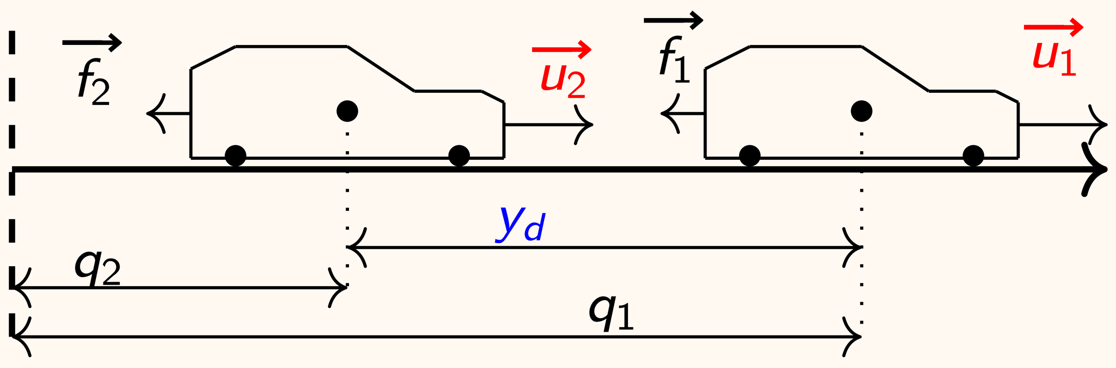
Topic very well understood from literature.

Particularity

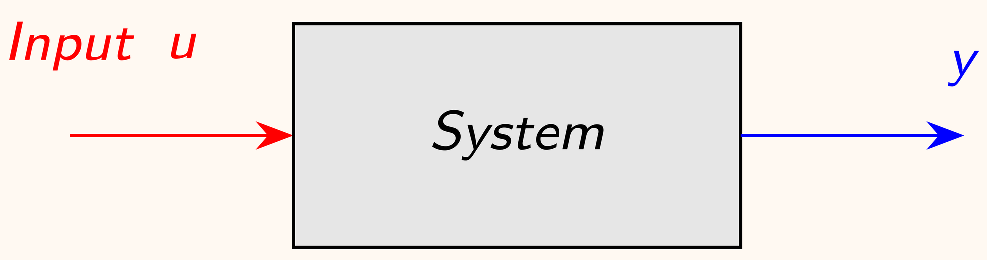
All the state variables are controlled.

Motivation

- What if you do not will to control all the state variables ?



- More generally



Questions:

- Can we reach any benchmark output y_{ref}?
- If YES, what is the suitable u to be used?

To answer these questions, Bertram and Sarachik introduced in [1] the notion of *Output controllability*.

Framework: LTI systems

We assume that the output is given by

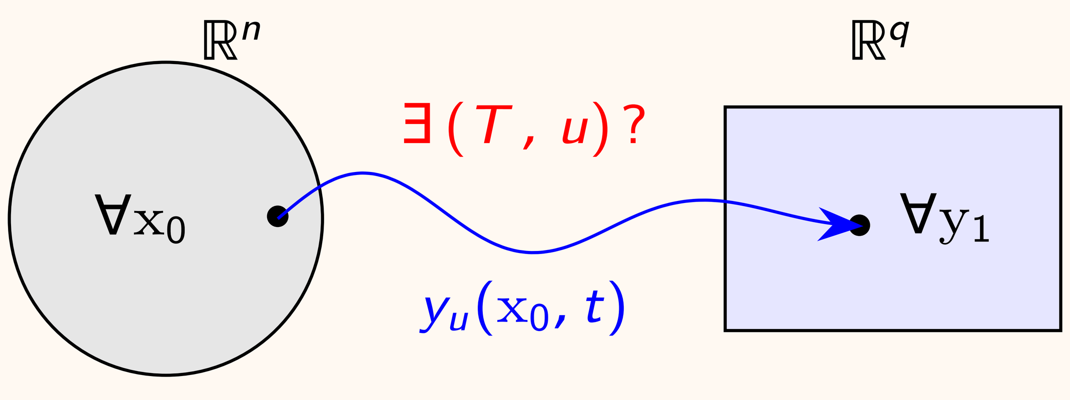
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \qquad (1)$$

$$x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad y(t) \in \mathbb{R}^q.$$

Discussed notions:

State to output controllability.

State to output controllability



u continuous and $y_u(x_0, T) = y_1$.

Two other notions of Output controllability can be found in [2]. In these notions, we discuss how to steer an initial output value to a prescribed output.

Theorem (Criteria)

The system (1) *is state to output controllable if and only if one of the following conditions is fulfilled*

(i) *There exists a time* $T > 0$ *such that the linear map* $E_T^o : \mathbf{C}^0([0,T];\mathbb{R}^m) \to \mathbb{R}^q$, *defined by* $E_T^o(u) = \int_0^T Ce^{(T-\tau)A}Bu(\tau)\mathrm{d}\tau + Du(T)$ *is surjective.*

(ii) $\mathrm{rk}\left[D \ CB \ CAB \ \cdots \ CA^{n-1}B\right] = q$. *(Given in [3])*

(iii) $\mathcal{K}_T := C\int_0^T e^{tA}BB^\top e^{tA^\top}\mathrm{d}t\, C^\top + DD^\top > 0$ *for some* $T > 0$. *(Also given in [3])*

(iv) $\mathrm{rk}\,(C|D) = q$ *and* $\mathrm{Im}\begin{pmatrix} C^\top \\ D^\top \end{pmatrix} \cap \left(\bigoplus_{\lambda\in\sigma(A)} E_\lambda \times \{0^m\} \right) = \{0\}$, *where*

$E_\lambda = \ker(A_\lambda^\top)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda - 1} \ker B^\top (A_\lambda^\top)^k\right)$, $A_\lambda = A - \lambda I_n$ *and* n_λ, *the algebraic multiplicity of* λ *in the minimal polynomial of* A.

(v) $\mathrm{rk}\,(C|D) = q$ *and* $\mathrm{rk}\begin{pmatrix} K_{\lambda_1} & 0 & \cdots & 0 & (C|D)^\perp \\ 0 & K_{\lambda_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & K_{\lambda_p} & (C|D)^\perp \end{pmatrix} = (n+m)p$, *where* $\{\lambda_1, \lambda_2, \cdots, \lambda_p\} = \sigma(A)$,

$K_\lambda = \begin{pmatrix} M_\lambda & 0 \\ 0 & I_m \end{pmatrix}$ *and* $M_\lambda = \left(A_\lambda^{n_\lambda} | A_\lambda^{n_\lambda - 1}B | \cdots | A_\lambda B | B\right)$.

(vi) $\mathcal{G}_T^o := \int_0^T H_o(T,t)H_o(T,t)^\top \mathrm{d}t > 0$ *for some* $T > 0$, *where* $H_o(T,t) = C\int_t^T e^{(T-\tau)A}B\mathrm{d}\tau + D$.

Theorem (Control computation)

Let $(x_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^q$, *and assume that system* (1) *is SOC.*
For every $T > 0$ *and* $u_0 \in \mathbb{R}^m$, *the control steering* x_0 *to* y_1 *in time* T *is*

$$u(t) = u_0 + \int_0^t H_o(T,\tau)^\top \mathrm{d}\tau (\mathcal{G}_T^o)^{-1}\left(y_1 - Ce^{TA}x_0 - H_o(T,0)u_0\right) \qquad (2)$$

Furthermore, this control is the unique minimizer of

$$\min \frac{1}{2}\int_0^T |\dot{u}(t)|_m^2 \mathrm{d}t$$
$$u \in H^1([0,T];\mathbb{R}^m),\ u(0) = u_0,\ y_1 = y_u(T, x_0).$$

Application to the cars example.

Consider for $t \geq 0$, system (1) with $f_\ell = -\alpha_\ell v_\ell$ where v_ℓ is the speed of car ℓ for $\ell = 1,2$ and m_ℓ stands for the mass of car ℓ. Choosing $m_1 = m_2 = 1$ and $\alpha_1 = \alpha_2 = 1/2$ the system is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

- One can see that this system is state to output controllable in any time $T > 0$.
- Goal: steer the system from $x_0 = (1,\ 0,\ 1,\ 0)^\top$ to $y_1 = (2,\ 0,\ 0)^\top$ in time $T = 1$.

Applying (2) with $u_0 = (1,\ 0)^\top$, we get

$$u = (u_1,\ u_2)^\top,\ u_1 = 1 - u_2,\ u_2(t) = at + bt^2 + c(e^{\frac{t}{2}} - 1) \text{ where } a = (254 + 98e - 316e^{\frac{1}{2}})/d,$$
$$b = (105 + 45e - 138e^{\frac{1}{2}})/d, \text{ and } c = (54 - 32e^{\frac{1}{2}})/d, \text{ with } d = 222e - 736e^{\frac{1}{2}} + 610$$

Using the matrix $\mathcal{K}_T$, we get $u = (u_1,\ u_2)^\top$, $u_2 = -u_1$ and $u_1(t) = (2e^{\frac{t}{2}} - e^{\frac{1}{2}} - 1)/(6e^{\frac{1}{2}} - 10)$ for $t \in [0,1)$ and $u(1) = 0$.

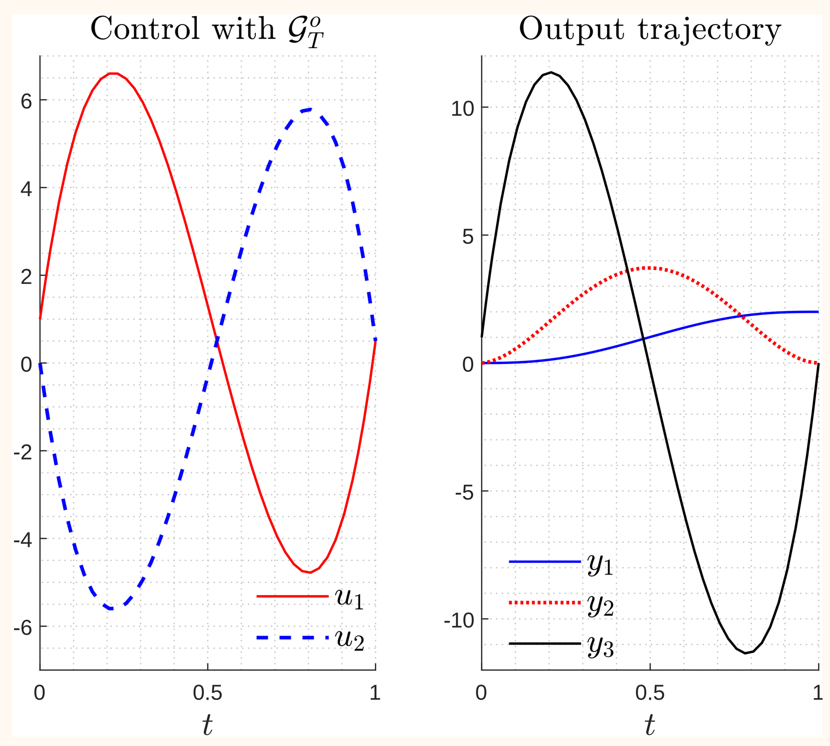


Figure: Continuous output trajectories with our matrix $\mathcal{G}_T^o$.

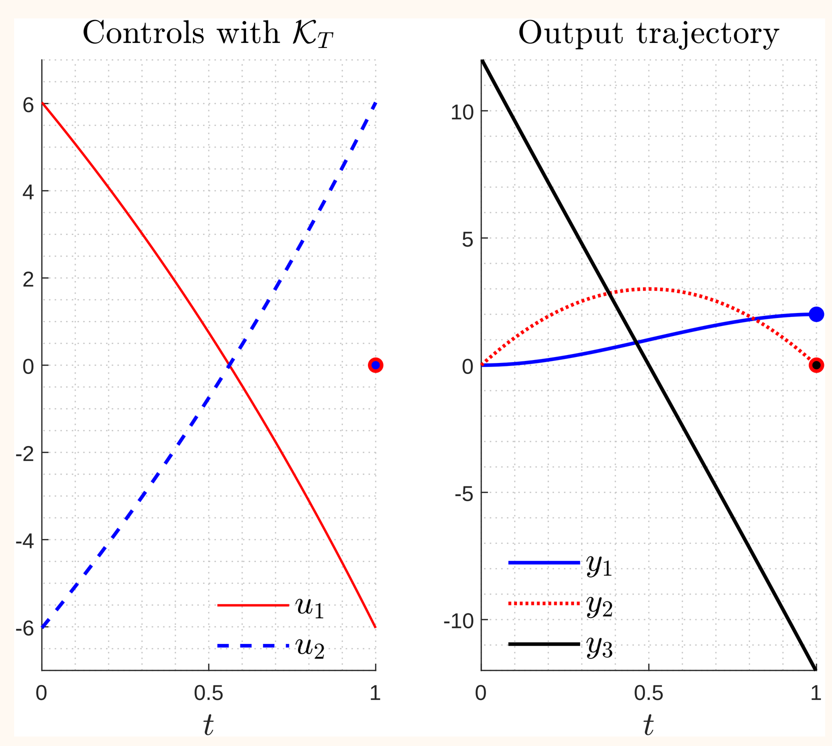


Figure: Discontinuous output trajectories with the matrix $\mathcal{K}_T$ proposed by Kreindler and Sarachik.

References

J. Bertram and P. Sarachik. "On optimal computer control". *IFAC Proceedings Volumes* 1.1 (1960).

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E. Kreindler and P. Sarachik. "On the concepts of controllability and observability of linear systems". *IEEE Transactions on Automatic Control* 9.2 (1964).