STABLE DETERMINATION OF COEFFICIENTS IN A COUPLED SEMILINEAR PARABOLIC SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract

This work [4] is devoted to deriving a Lipschitz stability estimate for interior and boundary potentials in a coupled semilinear parabolic system with dynamic boundary conditions, using only one distributed observation component. The proof relies on some new Carleman estimates for dynamic boundary conditions.

Model

We consider the following system of coupled parabolic equations with dynamic boundary conditions.

$$\begin{cases} \partial_t y = \operatorname{div}(A(x)\nabla y) + p_{11}(x)y + p_{12}(x)z + p_{13}(x)f(y,z), & \text{in } \Omega_T,\\ \partial_t z = \operatorname{div}(A(x)\nabla z) + p_{21}(x)y + p_{22}(x)z, & \text{in } \Omega_T,\\ \partial_t y_\Gamma = \operatorname{div}_\Gamma(D(x)\nabla_\Gamma y_\Gamma) - \partial_\nu^A y + q_{11}(x)y_\Gamma + q_{12}(x)z_\Gamma \\ \quad + q_{13}(x)g(y_\Gamma, z_\Gamma), & \text{on } \Gamma_T,\\ \partial_t z_\Gamma = \operatorname{div}_\Gamma(D(x)\nabla_\Gamma z_\Gamma) - \partial_\nu^A z + q_{21}(x)y_\Gamma + q_{22}(x)z_\Gamma, & \text{on } \Gamma_T,\\ y_\Gamma(t,x) = y_{|\Gamma}(t,x), \qquad z_\Gamma(t,x) = z_{|\Gamma}(t,x), & \text{on } \Gamma_T,\\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}), \qquad (z, z_\Gamma)|_{t=0} = (z_0, z_{0,\Gamma}), & \Omega\times\Gamma, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain of smooth boundary Γ, $\Omega_T = (0,T)\times\Omega$, $\Gamma_T = (0,T)\times\Gamma$. Here, $(y_0, y_{0,\Gamma}), (z_0, z_{0,\Gamma}) \in L^2(\Omega)\times L^2(\Gamma)$ are the initial states, and the potentials are such that $p_{ij} \in L^\infty(\Omega)$ and $q_{ij} \in L^\infty(\Gamma)$. The nonlinearities $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous with respect to the two variables. We assume that the diffusion matrices A and D are symmetric and uniformly elliptic. By $y_{|\Gamma}$, one designates the trace of y, and by $\partial_\nu^A y := (A\nabla y\cdot\nu)_{|\Gamma}$ the conormal derivative. The operator $\operatorname{div} = \operatorname{div}_x$ stands for the Euclidean divergence operator in Ω, and $\operatorname{div}_\Gamma$ stands for the tangential divergence operator in Γ.
Semilinear systems such as (1) arise in biological and ecological models in climatology, special flows in hydrodynamics, and chemical reactions.

A brief Literature

In the case of coupled systems with static boundary conditions, Cristofol et al. have proven in [3] some stability results for a coefficient in a nonlinear parabolic system. Their proof is based on a modified Carleman estimate with one observation component.
As for dynamic boundary conditions, Maniar et al. [2] have proven a Carleman estimate for a (single) heat system with dynamic boundary conditions in the isotropic case, i.e., $A = dI$ et $D = \delta I$, where $d, \delta > 0$ are positive constants.

Inverse Problem

For fixed constant $R > 0$, we denote the set of admissible potentials by

$$\mathcal{P} := \{(p,q) \in \mathbb{L}^\infty \colon \|p\|_\infty, \|q\|_\infty \le R\}. \tag{2}$$

We are interested in the simultaneous determination of the coupling coefficients using only one observation component, namely, the identification of the potentials

$$\mathfrak{p}_{13} := (p_{13}, q_{13}) \quad \text{and} \quad \mathfrak{p}_{21} := (p_{21}, q_{21})$$

belonging to $\mathcal{P}$, from the measurement $z|_{(t_0,t_1)\times\omega}$, $(t_0, t_1) \subset (0,T)$, $\omega \Subset \Omega$.

Results

We set $\mathbb{L}^2 := L^2(\Omega)\times L^2(\Gamma)$ and $\mathbb{H}^2 := \{(u, u_\Gamma) \in H^2(\Omega)\times H^2(\Gamma) : u_{|\Gamma} = u_\Gamma\}$.

Assumption I.

(i) $(p_{ij}, q_{ij}), (\widetilde{p}_{13}, \widetilde{q}_{13}), (\widetilde{p}_{21}, \widetilde{q}_{21}) \in \mathcal{P}$, for $i = 1,2$ and $j = 1,2,3$.

(ii) There exist constants $r > 0$ and $p_0 > 0$ such that

$$\begin{aligned} &\widetilde{y}_0, \widetilde{y}_{0,\Gamma} \ge r \quad \text{and} \quad \widetilde{z}_0, \widetilde{z}_{0,\Gamma} \ge 0,\\ &p_{11}r + p_{12}\widetilde{z}_0 + \widetilde{p}_{13}f(r, \widetilde{z}_0) \ge 0,\\ &q_{11}r + q_{12}\widetilde{z}_{0,\Gamma} + \widetilde{q}_{13}g(r, \widetilde{z}_{0,\Gamma}) \ge 0,\\ &p_{21} \ge p_0 \quad \text{and} \quad \widetilde{p}_{21} \ge p_0. \end{aligned}$$

Assumption II. We set $\theta = \frac{t_0+t_1}{2}$.

(i) $f, g \in W^{1,\infty}(\mathbb{R}^2)$.

(ii) $\exists r_1 > 0\colon\ |f(\widetilde{y}, \widetilde{z})(\theta,\cdot)| \ge r_1 > 0, \qquad |g(\widetilde{y}_\Gamma, \widetilde{z}_\Gamma)(\theta,\cdot)| \ge r_1 > 0.$

(iii) $\partial_t f(\widetilde{y}, \widetilde{z}) \in L^2(t_0, t_1; L^\infty(\Omega)), \qquad \partial_t g(\widetilde{y}_\Gamma, \widetilde{z}_\Gamma) \in L^2(t_0, t_1; L^\infty(\Gamma)).$

We mainly aim to establish the following Lipschitz stability estimate.

Theorem. *Let assumptions **Assumption I** and **Assumption II** be satisfied. We further assume that* $\widetilde{Y}_0, \widetilde{Z}_0 \in \mathbb{H}^2$ *and* $(y,z)(\theta,\cdot) = (\widetilde{y}, \widetilde{z})(\theta,\cdot)$ *in* Ω. *Then there exists a positive constant* $C = C(\Omega, \omega, p_0, \theta, t_0, t_1, r, R)$ *such that*

$$\|(p_{21} - \widetilde{p}_{21}, q_{21} - \widetilde{q}_{21})\|_{\mathbb{L}^2} + \|(p_{13} - \widetilde{p}_{13}, q_{13} - \widetilde{q}_{13})\|_{\mathbb{L}^2} \le C\|\partial_t z - \partial_t \widetilde{z}\|_{L^2(\omega_{t_0,t_1})}.$$

The proof draws on the following steps:

1. Positivity of the solution: We consider the solution of the following system.

$$\begin{cases} \partial_t y = \operatorname{div}(A(x)\nabla y) + f_1(y,z), & \text{in } \Omega_T,\\ \partial_t z = \operatorname{div}(A(x)\nabla z) + f_2(y,z), & \text{in } \Omega_T,\\ \partial_t y_\Gamma = \operatorname{div}_\Gamma(D(x)\nabla_\Gamma y_\Gamma) - \partial_\nu^A y + g_1(y_\Gamma, z_\Gamma), & \text{on } \Gamma_T,\\ \partial_t z_\Gamma = \operatorname{div}_\Gamma(D(x)\nabla_\Gamma z_\Gamma) - \partial_\nu^A z + g_2(y_\Gamma, z_\Gamma), & \text{on } \Gamma_T,\\ y_\Gamma(t,x) = y_{|\Gamma}(t,x), \qquad z_\Gamma(t,x) = z_{|\Gamma}(t,x), & \text{on } \Gamma_T,\\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}), \qquad (z, z_\Gamma)|_{t=0} = (z_0, z_{0,\Gamma}), & \Omega\times\Gamma. \end{cases} \tag{3}$$

We will use the following assumption to prove that (3) has nonnegative solution for nonnegative initial data:
(**QP**) The functions f_1, f_2, g_1 and g_2 are quasi-positive. That is,

$$\begin{aligned} f_1(0,v) \ge 0 \quad &\text{and} \quad g_1(0,v) \ge 0 \quad \forall v \ge 0,\\ f_2(u,0) \ge 0 \quad &\text{and} \quad g_2(u,0) \ge 0 \quad \forall u \ge 0. \end{aligned}$$

Lemma. Let $(y_0, y_{0,\Gamma})$ and $(z_0, z_{0,\Gamma})$ be componentwise nonnegative initial data. Suppose that (**QP**) holds true. Then the solution $(y, z, y_\Gamma, z_\Gamma)$ of (3) is componentwise nonnegative.

2. General Carleman estimate: We adopt the same weight functions α and ξ as in [1]. Let $\tau \in \mathbb{R}$. Denote

$$\begin{aligned} I_\Omega(\tau, z) = &\int_{\Omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau-1}\left(|\partial_t z|^2 + |\operatorname{div}(A(x)\nabla z)|^2\right) \mathrm{d}x\,\mathrm{d}t + \lambda^2 \int_{\Omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau+1}|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t\\ &+ \lambda^4 \int_{\Omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau+3}|z|^2 \,\mathrm{d}x\,\mathrm{d}t,\\ I_\Gamma(\tau, z_\Gamma, z) = &\int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau-1}\left(|\partial_t z_\Gamma|^2 + |\operatorname{div}_\Gamma(D(x)\nabla_\Gamma z_\Gamma)|^2\right) \mathrm{d}S\,\mathrm{d}t + \lambda \int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau+1}|\nabla_\Gamma z_\Gamma|^2 \,\mathrm{d}S\,\mathrm{d}t\\ &+ \lambda^3 \int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau+3}|z_\Gamma|^2 \,\mathrm{d}S\,\mathrm{d}t + \lambda \int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau+1}|\partial_\nu^A z|^2 \,\mathrm{d}S\,\mathrm{d}t. \end{aligned}$$

Lemma (Carleman estimate). Let $\tau \in \mathbb{R}$. There are three positive constants $\lambda_1 = \lambda_1(\Omega, \omega)$, $s_1 = s_1(\Omega, \omega, \tau)$ and $C = C(\Omega, \omega, \tau)$ such that, for any $\lambda \ge \lambda_1$ and $s \ge s_1$, the following inequality holds

$$\begin{aligned} I_\Omega(\tau, z) + I_\Gamma(\tau, z_\Gamma, z) \le C \Bigg[&\lambda^4 \int_{\omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau+3}|z|^2 \,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\Omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau}|Lz|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}(s\xi)^{\tau}|L_\Gamma(z_\Gamma, z)|^2 \,\mathrm{d}S\,\mathrm{d}t \Bigg] \end{aligned}$$

for all $Z = (z, z_\Gamma) \in H^1\left(0, T; \mathbb{L}^2\right) \cap L^2\left(0, T; \mathbb{H}^2\right)$.

Remark. The above Carleman estimate does not yield the desired stability estimate for the nonlinear coupled system (1), since we need appropriate powers in s and λ to absorb some terms on the right-hand side. To this end, we need a modified form of Carleman estimate with one observation.

3. A modified Carleman estimate: Consider the following system.

$$\begin{cases} \partial_t y = \operatorname{div}(A(x)\nabla y) + p_{11}(x)y + p_{12}(x)z + f_1, & \text{in } \Omega_T,\\ \partial_t z = \operatorname{div}(A(x)\nabla z) + p_{21}(x)y + p_{22}(x)z + f_2, & \text{in } \Omega_T,\\ \partial_t y_\Gamma = \operatorname{div}_\Gamma(D(x)\nabla_\Gamma y_\Gamma) - \partial_\nu^A y + q_{11}(x)y_\Gamma + q_{12}(x)z_\Gamma + g_1, & \text{on } \Gamma_T,\\ \partial_t z_\Gamma = \operatorname{div}_\Gamma(D(x)\nabla_\Gamma z_\Gamma) - \partial_\nu^A z + q_{21}(x)y_\Gamma + q_{22}(x)z_\Gamma + g_2, & \text{on } \Gamma_T,\\ y_\Gamma(t,x) = y_{|\Gamma}(t,x), \qquad z_\Gamma(t,x) = z_{|\Gamma}(t,x), & \text{on } \Gamma_T,\\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}), \qquad (z, z_\Gamma)|_{t=0} = (z_0, z_{0,\Gamma}), & \Omega\times\Gamma. \end{cases} \tag{4}$$

Lemma. There exist three positive constants $\lambda_1 = \lambda_1(\Omega, \omega) \ge 1$, $s_1 = s_1(T, \lambda_1) > 1$ and $C = C(\Omega, \omega, R, T, r_0)$ such that, for any $\lambda \ge \lambda_1$ and $s \ge s_1$ with fixed $\epsilon > 0$, the following inequality holds

$$\begin{aligned} &\lambda^{-4+\epsilon}[I_\Omega(-3, y) + I_\Gamma(-3, y_\Gamma, y)] + I_\Omega(0, z) + I_\Gamma(0, z_\Gamma, z)\\ &\le C \Bigg[s^4\lambda^{4+\epsilon} \int_{\omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}\xi^4|z|^2 \,\mathrm{d}x\,\mathrm{d}t\\ &\quad + s^{-3}\lambda^{-9+\epsilon}\left(\int_{\Omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}\xi^{-3}|f_1|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}\xi^{-3}|g_1|^2 \,\mathrm{d}S\,\mathrm{d}t \right)\\ &\quad + \lambda^{2\epsilon}\left(\int_{\Omega_{t_0,t_1}} \mathrm{e}^{-2s\alpha}|f_2|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_{t_0,t_1}} \mathrm{e}^{-2s\alpha}|g_2|^2 \,\mathrm{d}S\,\mathrm{d}t \right) \Bigg]. \end{aligned}$$

References

[1] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of Evolution Equations.* Vol. 34. 1996.

[2] L. Maniar M. Meyries and R. Schnaubelt. "Null controllability for parabolic equations with dynamic boundary conditions of reactive-diffusive type". In: *Evol. Equat. and Cont. Theo.* 6 (2017), pp. 381–407.

[3] M. Cristofol P. Gaitan H. Ramoul and M. Yamamoto. "Identification of two coefficients with data of one component for a nonlinear parabolic system". In: *Applicable Analysis* 91 (Mar. 2012), pp. 2073–2081.

[4] E. M. Ait Ben Hassi S. E. Chorfi and L. Maniar. "Stable determination of several coefficients by one observation in a coupled semilinear parabolic system with dynamic boundary conditions". In: (2021).

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