Controllability of a bilinear Schrödinger equation by a power series expansion

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Sussman's example

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2^2 + x_1^3. \end{cases} \quad (1)$$

The system (1) is controllable on $\mathrm{Span}((1,0,0),(0,1,0))$. Denote by u_i the i-th primitive of u vanishing at zero. What about the third component

$$x_3(T;\ u,\ 0) = \int_0^T u_2(t)^2 dt + \int_0^T u_1(t)^3 dt \quad ?$$

Definition: E-STLC of (1)

(1) is *E*-STLC if for all $T>0$ and $\varepsilon>0$, there exists $\delta>0$ such that for all data $|x_0|+|x_f|<\delta$, **there exists** $u \in E_T$ **with** $\|u\|_{E_T}<\varepsilon$ such that

$$x(T;\ u,\ x_0) = x_f.$$

No $W^{1,\infty}$ -STLC

Regular controls ⇝ quadratic wins!

Theorem

The system (1) **is not** $W^{1,\infty}$**-STLC**: there exists $A, T^*>0$ s. t. for all $T\in(0,T^*)$, there exists $\varepsilon>0$ s. t. **for all** $u\in W^{1,\infty}(0,T)$ **with** $\|u\|_{W^{1,\infty}(0,T)}<\varepsilon$,

$$x_3(T;\ u,\ 0) \geq A\int_0^T u_2(t)^2 dt > 0.$$

By integrations by parts, when $\|u'\|_{L^\infty(0,T)}$ is small,

$$\int_0^T u_1(t)^3 dt = \int_0^T u_2(t)^2 u'(t) dt = o\left(\int_0^T u_2(t)^2 dt\right).$$

So, if $\|u'\|_{L^\infty(0,T)}$ is small enough,

$$x_3(T;\ u,\ 0) \geq \left(1-\|u'\|_{L^\infty(0,T)}\right)\int_0^T u_2(t)^2 dt > 0.$$

L^∞-STLC

Theorem (Sussman, 1983)

The system (1) **is** L^∞**-STLC**.

The **cubic** term **wins** for controls of the form:

$$u_\lambda(t) = \sqrt{\lambda}\phi''\left(\frac{t}{\lambda}\right), \quad \lambda>0.$$

Size of the controls:

$$\|u_\lambda\|_{L^\infty(0,T)} \approx \sqrt{\lambda} \ll 1, \quad \|u_\lambda'\|_{L^\infty(0,T)} \approx \frac{1}{\sqrt{\lambda}} \gg 1.$$

Then, the third component of (1) is given by

$$x_3(T;\ u_\lambda,\ 0) = \lambda^2\int_0^1 \phi'(\theta)^3 d\theta + \lambda^6\int_0^1 \phi(\theta)^2 d\theta.$$

Taking $\lambda = |a|^{\frac{2}{11}}$ and $\int_0^1 \phi'(\theta)^3 d\theta = \mathrm{sign}(a)$,

$$x_3(T;\ u_a,\ 0) = a + o(a).$$

Less regular controls ⇝ cubic wins!

Schrödinger equation

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T). \end{cases} \quad \text{(Schro)}$$

This is a bilinear control system where the **state** is ψ and u denotes a scalar **control**.

Definition: STLC around the ground state in X with controls in E

(Schro) is **STLC** if for all $T>0$ and $\varepsilon>0$, there exists $\delta>0$ such that for all (ψ_0,ψ_f) in X with $\|\psi_0-\varphi_1\|_X<\delta$ and $\|\psi_f-\varphi_1 e^{-i\lambda_1 T}\|_X<\delta$, **there exists** $u\in L^2(0,T)\cap E_T$ **with** $\|u\|_{E_T}<\varepsilon$ such that

$$\psi(T;\ u,\ \psi_0) = \psi_f.$$

To study the STLC of (Schro), perform a **power series expansion**,

$$\psi(t) \approx \varphi_1 e^{-i\lambda_1 t} + \mathrm{Lin} + \mathrm{Quad} + \mathrm{Cub} + \cdots.$$

The first-order term Ψ

$$\begin{cases} i\partial_t\Psi(t,x) = -\partial_x^2\Psi(t,x) - u(t)\mu(x)\varphi_1(x)e^{-i\lambda_1 t}, & (t,x)\in(0,T)\times(0,1), \\ \Psi(t,0)=\Psi(t,1)=0, & t\in(0,T). \end{cases} \quad \text{(Lin)}$$

The solution can be computed explicitly as

$$\Psi(T;\ u,\ 0) = i\sum_{j=1}^{+\infty}\langle\mu\varphi_1,\varphi_j\rangle\int_0^T u(t)e^{i(\lambda_j-\lambda_1)t}dt\,\varphi_j e^{-i\lambda_j T}.$$

Proposition: Controllability of the linearized equation (Lin)

Let $(p,k)\in\mathbb{N}^2$ and J a subset of $\mathbb{N}^*$. If μ is in $H^{2(p+k)+3}((0,1),\mathbb{R})$ with $\mu^{(2n+1)}(0)=\mu^{(2n+1)}(1)=0$ for all $n=0,\dots,p-1$ such that there exists $c>0$ such that

$$\forall j\in J, \quad |\langle\mu\varphi_1,\varphi_j\rangle| \geq \frac{c}{j^{2p+3}}, \quad (2)$$

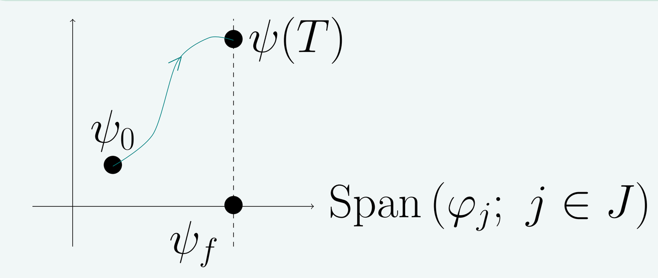
then (Lin) is **controllable** in projection in $H^{2(p+k)+3}_{(0)}(0,1)$ with controls in $H^k_0(0,T)$ with the **same control map.**

STLC in projection

(Lin) controllable ⇒ (Schro) STLC

Theorem: STLC in projection of (Schro)

Let μ in $H^{2(p+k)+3}((0,1),\mathbb{R})$ with $\mu^{(2n+1)}(0)=\mu^{(2n+1)}(1)=0$ for all $n=0,\dots,p-1$ and satisfying (2). Then, (Schro) is **STLC in projection** around the ground state in $H^{2(p+m)+3}_{(0)}(0,1)$ with controls in $H^m_0(0,T)$ for every $m\in\{0,\dots,k\}$ with the **same control map**.



Controllability along φ_K when $\langle\mu\varphi_1,\varphi_K\rangle=0$?

References

[1] M. Bournissou. *Local controllability of the bilinear 1D Schrödinger equation with simultaneous estimates.* arXiv : 2107.08817, 2021.

[2] M. Bournissou. *Quadratic behaviors of the 1D linear Schrödinger equation with bilinear control.* arXiv : 2111.01476, 2021.

[3] K. Beauchard and M. Morancey. *Local controllability of 1D Schrödinger equations with bilinear control and minimal time.* Math. Control Relat. Fields, 2014.

The second-order term ξ

$$\begin{cases} i\partial_t\xi(t,x) = -\partial_x^2\xi(t,x) - u(t)\mu(x)\Psi(t,x), & (t,x)\in(0,T)\times(0,1), \\ \xi(t,0)=\xi(t,1)=0, & t\in(0,T). \end{cases} \quad \text{(Quad)}$$

One can compute

$$\langle\xi(T;\ u,\ 0),\varphi_K e^{-i\lambda_1 T}\rangle = \int_0^T u(t)\int_0^t u(\tau)h_T(t,\tau)d\tau dt.$$

Proposition

$u\mapsto\langle\mathrm{Quad},\varphi_K\rangle$ is coercive?

If h_T is in $C^{2n}(\mathbb{R}^2,\mathbb{C})$, then for all $u\in L^1(0,T)$,

$$\int_0^T u(t)\int_0^t u(\tau)h_T(t,\tau)d\tau dt = -i\sum_{p=1}^n A_K^p\int_0^T u_p(t)^2 e^{i(\lambda_K-\lambda_1)(t-T)}dt + (\dots).$$

Proposition: Coercivity of the quadratic term

If $A_K^1=\dots=A_K^{n-1}=0$ and $A_K^n\neq0$, then there exists $T^*>0$ such that for all $T\in(0,T^*)$,

$$-\mathrm{sign}(A_K^n)\Im\langle\xi(T),\varphi_K e^{-i\lambda_1 T}\rangle \geq \frac{|A_K^n|}{4}\int_0^T u_n(t)^2 dt + (\dots).$$

Quadratic obstructions

Proposition: estimate of the cubic remainder

For regular controls, $\mathrm{Cub}=o(\mathrm{Quad})$

There exists $C>0$ such that for all $u\in H^{2n-3}(0,T)$,

$$\langle(\psi-\psi_1-\Psi-\xi)(T),\varphi_K e^{-i\lambda_1 T}\rangle \leq C\|u_1\|_{L^2}^3 \leq C\left(\|u^{(2n-3)}\|_{L^2} + T^{-2n+3}\|u\|_{L^2}\right)\|u_n\|_{L^2}^2.$$

Theorem: No STLC for regular controls because of drifts

Let $(K,n)\in\mathbb{N}^{*2}$. Assume that μ satisfies Hyp(K,n). If $n\geq2$ (resp. $n=1$), then (Schro) is **not** H^{2n-3}**-STLC** (resp. **not** $W^{-1,\infty}$**-STLC**): there exists $C,A,T^*>0$ such that for all $T\in(0,T^*)$, there exists $\varepsilon>0$ such that **for all** $u\in H^{2n-3}(0,T)$ (resp. $u\in L^2(0,T)$) **with** $\|u\|_{H^{2n-3}(0,T)}\leq\varepsilon$ (resp. $\|u_1\|_{L^\infty(0,T)}\leq\varepsilon$), then the solution satisfies

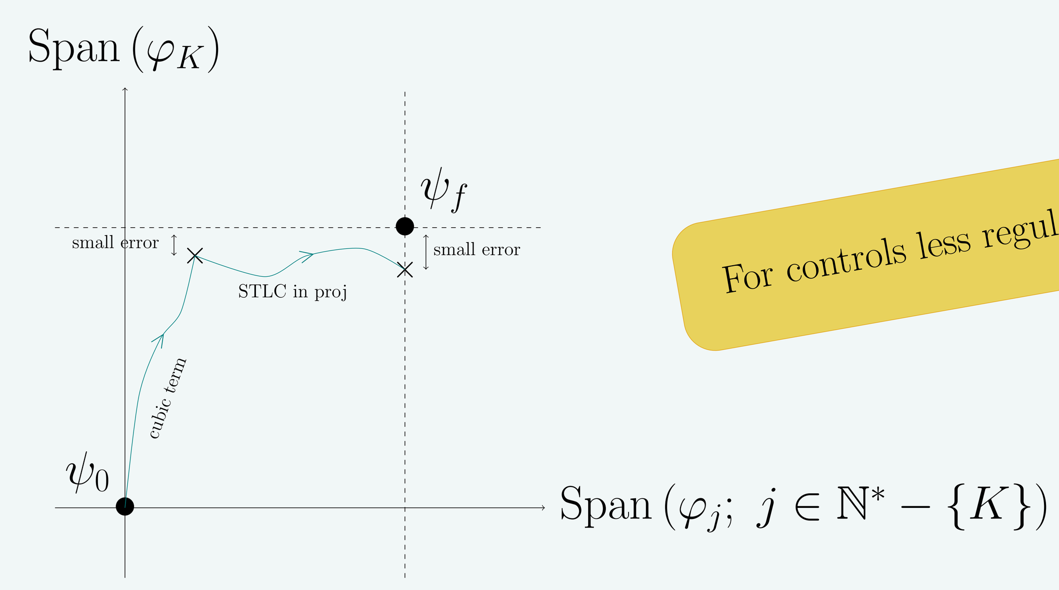
$$-\mathrm{sign}(A_K^n)\Im\langle\psi(T;\ u,\ \varphi_1),\varphi_K e^{-i\lambda_1 T}\rangle \leq A\|u_n\|^2_{L^2(0,T)} - C\|(\psi-\psi_1)(T)\|^2_{L^2(0,1)}.$$

Positive STLC result despite a drift!

Theorem

In the setting of the third drift,

- (Schro) is not H^3-STLC because of a **drift** quantified by the H^{-3} norm of the control.
- But (Schro) is H^2-STLC thanks to the **cubic** term **despite a quadratic drift**.



For controls less regular, $\mathrm{Quad}=o(\mathrm{Cub})$