ISS Lyapunov strictification via observer design and integral action control for a Korteweg-de Vries equation

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Motivation

The aim of this work is to use an integral action and an output feedback control law to solve to the output regulation problem of a linear Korteweg-de-Vries (KdV) system subject to a distributed disturbance.

Problem statement

Consider the Korteweg-de Vries equation

$$\begin{cases} w_t + w_x + w_{xxx} = d(x), \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+ \\ w_x(t,L) = u(t), \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L], \end{cases} \tag{1}$$

$u(t) \in \mathbb{R}$ is a **control** $d \in L^2(0,L)$ is an unknown **disturbance**.

Question

Assume the **output** is $y(t) := w_x(t,0)$. Is it possible to design an **output feedback law** $u(t) := f(w_x(t,0))$ such that:

$$\lim_{t\to+\infty} |y(t) - r| = 0,$$

where r is a given **reference**, and despite the disturbance ?

A finite-dimensional example

$$\dot w(t) = u(t) + d,$$

$w(t) \in \mathbb{R}$, u is the **control**, d is a constant **disturbance**.
How can one design a feedback $u(t) = f(w(t))$ such that

$$\lim_{t\to+\infty} |w(t) - r| = 0, \quad \text{where } r \text{ is a given reference ?}$$

Obstruction

Static feedback-laws are **not enough**. Indeed, if $u(t) = -k(w(t) - r)$, then one has

$$|w(t) - r|^2 \leq |w(0) - r)|^2 e^{-(k-\epsilon)t} + \frac{1 - e^{-(k-\epsilon)t}}{\epsilon(k-\epsilon)} d^2$$

with $k > 0$ and $\epsilon > 0$ such that $k - \epsilon > 0$. This mean that the feedback u is not robust with respect to the disturbance d.

A simple solution known for a long time to this problem is to add an integral term which also uses the information of the previous disturbance and absorbs this disturbance.

The integral action principle

PI controller

$$\begin{cases} \dot w(t) = -\underbrace{k_p(w(t)}_{\text{proportional action}} - \underbrace{k_i z(t)}_{\text{integral action}} + d, \\ \dot z(t) = w(t) - r \end{cases}$$

- The **proportional** action **stabilizes** w.
- The **integral** action modifies the **equilibrium points**.

Stability and Observability properties

Assuming that $L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}} \,|\, k, l \in \mathbb{N} \right\}$, then, when $u = 0$ and $d = 0$ one has

1. the origin of (1) is **globally exponentially stable** [Rosier, 1997]
2. the output $y(t) = w_x(t,0)$ is **exactly observable** [Rosier, 1997].

Main result

The open-loop is stable ⇒ **no need** of a **proportional** action.

$$\begin{cases} w_t + w_x + w_{xxx} = d(x), \ (t,x) \in \mathbb{R}_+, \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = kz(t), \ t \in \mathbb{R}_+ \\ \dot z(t) = y(t) - r, \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ z(0) = z_0, \ x \in [0,L]. \end{cases}$$

Main result

Notation: w_∞, z_∞ are the equilibrium points.
$D(\mathcal{A}) := \{z, w \in \mathbb{R} \times L^2(0,L) \mid w(0) = w(L) = 0, \ w'(L) = z\}$.

Theorem

Let $k \in (0, k^*)$. Then, for any $(d, r) \in L^2(0,L) \times \mathbb{R}$:

1. There exist $\nu, C > 0$ such that, for all $(z_0, w_0) \in \mathbb{R} \times L^2(0,L)$, and for all $t \geq 0$

$$\|(z,w) - (z_\infty, w_\infty)\|_{L^2\times\mathbb{R}} \leq Ce^{-\nu t}\|(z_0,w_0) - (z_\infty, w_\infty)\|_{L^2\times\mathbb{R}}$$

2. The output y is **regulated** towards the reference r. In other words, for any $(z_0, w_0) \in D(\mathcal{A})$

$$\lim_{t\to+\infty} |w_x(t,0) - r| = 0.$$

for any **strong solution**.

Proof

I. Build a **ISS-Lyapunov functional** for the KdV equation.
When $u = d = 0$, we recall that $E(w) := \frac{1}{2}\|w\|^2_{L^2}$ satisfies:

$$\frac{d}{dt}E(w) = -|w_x(t,0)|^2 = -|y(t)|^2.$$

Then, **nonpositivity** is ensured but the right hand side **depends only on the output**.

⇒ It is a **weak Lyapunov functional**.
Following [Praly, 2019], we **strictify** E with an **observer**.

Strictification ?

Consists in **modifying** a weak Lyapunov functional to make it **strict** ([Malisoff & Mazenc, 2009], [Prieur & Mazenc, 2012]).

Using the bacsktepping method based on the Fredholm transform, one proves the existence of $p \in L^2(0,L)$ such that the observer $\hat w$

$$\begin{cases} \hat w_t + \hat w_x + \hat w_{xxx} + p(x)[y(t) - \hat w_x(t,0)] = 0 \\ \hat w(t,0) = \hat w(t,L) = \hat w_x(t,L) = 0. \end{cases}$$

converge to w in the nominal condition. Consider $\tilde w := w - \hat w$ which satisfies

$$\begin{cases} \tilde w_t + \tilde w_x + \tilde w_{xxx} + p(x)\tilde w_x(t,0) = d(x) \\ \tilde w(t,0) = \tilde w(t,L) = 0, \ \hat w_x(t,L) = u(t). \end{cases} \tag{2}$$

One proves the existence of a ISS Lyapunov functional U for (2). Then

$$\frac{d}{dt}U(\tilde w) \leq -\lambda U(\tilde w) + |u(t)|^2 + \|d\|^2_{L^2}.$$

We rewrite (1) as follows

$$\begin{cases} w_t + w_x + w_{xxx} + p(x)w_x(t,0) - p(x)w_x(t,0) = d(x) \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = u(t). \end{cases}$$

Then, one has

$$\frac{d}{dt}U(w) \leq -\lambda U(w) + |u(t)|^2 + 2\|p\|^2_{L^2}|w_x(t,0)|^2 + 2\|d\|^2_{L^2}.$$

Recall that

$$\frac{d}{dt}E(w) \leq E(w) - |w_x(t,0)|^2 + |u(t)|^2 + \frac{1}{2}\|d\|^2_{L^2},$$

then choosing $V(w) := E(w) + aU(w)$ with $a = \frac{1}{2\|p\|^2_{L^2}}$, one has

$$\frac{d}{dt}V(w) \leq -\tilde\lambda U(w) + \tilde\sigma|u(t)|^2 + \gamma\|d\|^2_{L^2}$$

II. Use the **forwarding** method [Mazenc & Praly, 1996], based on a **linear operator** $\mathcal{M} : L^2(0,L) \to \mathbb{R}$:

$$W(w,z) = V(w) + b|z - \mathcal{M}w|^2.$$

III. Select the **gains**.

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